

# COBORDISMS OF MAPS WITH SINGULARITIES OF A GIVEN CLASS

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**ABSTRACT.** Let  $P$  be a smooth manifold of dimension  $p$ . We will describe the group of all cobordism classes of smooth maps of  $n$ -dimensional closed manifolds into  $P$  with singularities of a given class (including all fold singularities if  $n \geq p$ ) in terms of certain stable homotopy groups by applying the homotopy principle on the existence level, which is assumed to hold for those smooth maps. We will also deal with the oriented version and construct a classifying space of this oriented cobordism group in the dimensions  $n < p$  and  $n \geq p \geq 2$ .

## 1. INTRODUCTION

Let  $N$  and  $P$  be smooth ( $C^\infty$ ) manifolds of dimensions  $n$  and  $p$  respectively. Let  $k \gg n, p$  ( $k$  may be  $\infty$ ). Let  $J^k(N, P)$  denote the  $k$ -jet bundle of the manifolds  $N$  and  $P$  with the canonical projection  $\pi_N^k \times \pi_P^k$  onto  $N \times P$  and the fiber  $J^k(n, p)$  of all  $k$ -jets of smooth map germs  $(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ . Here,  $\pi_N^k$  and  $\pi_P^k$  map a  $k$ -jet to its source and target respectively. Let  $\Omega = \Omega(n, p)$  denote a nonempty open subspace of  $J^k(n, p)$  which is invariant with respect to the action of  $L^k(p) \times L^k(n)$ , where  $L^k(m)$  denotes the group of  $k$ -jets of germs of diffeomorphisms of  $(\mathbb{R}^m, 0)$ . Let  $\Omega(N, P)$  denote the open subbundle of  $J^k(N, P)$  associated to  $\Omega(n, p)$ . A smooth map  $f : N \rightarrow P$  is called an  $\Omega$ -regular map if and only if  $j^k f(N) \subset \Omega(N, P)$ .

Let  $C_\Omega^\infty(N, P)$  denote the space of all  $\Omega$ -regular maps of  $N$  to  $P$  equipped with the  $C^\infty$  topology. Let  $\Gamma_\Omega(N, P)$  denote the space consisting of all continuous sections of the fiber bundle  $\pi_N^k|_{\Omega(N, P)} : \Omega(N, P) \rightarrow N$  equipped with the compact-open topology. Then we have the continuous map  $j_\Omega : C_\Omega^\infty(N, P) \rightarrow \Gamma_\Omega(N, P)$  defined by  $j_\Omega(f) = j^k f$ . We say that  $\Omega(n, p)$  satisfies *the homotopy principle* (simply *h-principle*) if any section  $s$  in  $\Gamma_\Omega(N, P)$  has an  $\Omega$ -regular map  $f$  such that  $j^k f$  is homotopic to  $s$  as sections. In this paper we say that  $\Omega(n, p)$  satisfies *the relative homotopy principle on the existence level* if the following property (h-P) holds.

(h-P): *Let  $C$  be a closed subset of  $N$ . Let  $s$  be a section of  $\Gamma_\Omega(N, P)$  which has an  $\Omega$ -regular map  $g$  defined on a neighborhood of  $C$  into  $P$ , where  $j^k g = s$ . Then there exists an  $\Omega$ -regular map  $f : N \rightarrow P$  such that  $j^k f$  is homotopic to  $s$  relative to a neighborhood of  $C$  by a homotopy  $s_\lambda$  in  $\Gamma_\Omega(N, P)$  with  $s_0 = s$  and  $s_1 = j^k f$ .*

Let  $i_{+m} : J^k(n, p) \rightarrow J^k(n+m, p+m)$  denote the map defined by  $i_{+m}(j_0^k f) = j_0^k(f \times id_{\mathbb{R}^m})$  for a positive integer, where  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  and  $id_{\mathbb{R}^m}$  is the identity of  $\mathbb{R}^m$ . Let  $\Omega_\star = \Omega_\star(n+1, p+1)$  denote a nonempty open subspace of  $J^k(n+1, p+1)$  which is invariant with respect to the action of  $L^k(p+1) \times L^k(n+1)$  and satisfies  $i_{+1}(\Omega(n, p)) \subset \Omega_\star(n+1, p+1)$ . Let  $P$  be a smooth manifold of dimension  $p$ . We define the notion of  $\Omega_\star$ -cobordisms of  $\Omega$ -regular maps to  $P$ .

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Let  $f_i : N_i \rightarrow P$  ( $i = 0, 1$ ) be two  $\Omega$ -regular maps, where  $N_i$  are closed smooth  $n$ -dimensional manifolds. We say that they are  $\Omega_\star$ -cobordant when there exists an  $\Omega_\star$ -regular map, called an  $\Omega_\star$ -cobordism,  $\mathfrak{C} : (W, \partial W) \rightarrow (P \times [0, 1], P \times 0 \cup P \times 1)$  such that, for a sufficiently small positive real number  $\epsilon$ ,

(i)  $W$  is a compact smooth manifold of dimension  $n + 1$  with  $\partial W$  being  $N_0 \cup N_1$  and the collar of  $\partial W$  is identified with  $N_0 \times [0, \epsilon] \cup N_1 \times [1 - \epsilon, 1]$ ,

(ii)  $\mathfrak{C}|_{N_0 \times [0, \epsilon]} = f_0 \times id_{[0, \epsilon]}$  and  $\mathfrak{C}|_{N_1 \times [1 - \epsilon, 1]} = f_1 \times id_{[1 - \epsilon, 1]}$ .

We similarly define the notion of *oriented*  $\Omega_\star$ -cobordisms of  $\Omega$ -regular maps by providing manifolds concerned with orientations, where  $N_0 \cup N_1$  in (i) should be replaced by  $N_0 \cup (-N_1)$ . Let  $\mathfrak{N}(n, P; \Omega, \Omega_\star)$  (respectively  $\mathfrak{O}(n, P; \Omega, \Omega_\star)$ ) denote the monoid of all  $\Omega_\star$ -cobordism (respectively oriented  $\Omega_\star$ -cobordism) classes of  $\Omega$ -regular maps to  $P$ . In this paper we will describe these monoids of cobordism classes in terms of certain stable homotopy groups.

We need some notations for this purpose. Let  $\mathcal{E} \rightarrow X$  and  $\mathcal{F} \rightarrow Y$  be smooth vector bundles of dimensions  $n$  and  $p$  over smooth manifolds, and let  $\pi_X$  and  $\pi_Y$  be the projections of  $X \times Y$  onto  $X$  and  $Y$  respectively. Define the vector bundle  $J^k(\mathcal{E}, \mathcal{F})$  over  $X \times Y$  by

$$(1.1) \quad J^k(\mathcal{E}, \mathcal{F}) = \bigoplus_{i=1}^k \text{Hom}(S^i(\pi_X^*(\mathcal{E})), \pi_Y^*(\mathcal{F}))$$

with the canonical projections  $\pi_X^k : J^k(\mathcal{E}, \mathcal{F}) \rightarrow X$  and  $\pi_Y^k : J^k(\mathcal{E}, \mathcal{F}) \rightarrow Y$ . Here,  $S^i(\mathcal{E})$  is the vector bundle  $\bigcup_{x \in X} S^i(\mathcal{E}_x)$  over  $X$ , where  $S^i(\mathcal{E}_x)$  denotes the  $i$ -fold symmetric product of the fiber  $\mathcal{E}_x$  over  $x$ . The canonical fiber  $\bigoplus_{i=1}^k \text{Hom}(S^i(\mathbb{R}^n), \mathbb{R}^p)$  is canonically identified with  $J^k(n, p)$ . If we provide  $N$  and  $P$  with Riemannian metrics, then  $J^k(TN, TP)$  is identified with  $J^k(N, P)$  over  $N \times P$  (see Section 2). Let  $\Omega(\mathcal{E}, \mathcal{F})$  denote the open subbundle of  $J^k(\mathcal{E}, \mathcal{F})$  associated to  $\Omega(n, p)$ .

Let  $G_m$  refer to the grassmann manifold  $G_{m, \ell}$  (respectively oriented grassmann manifold  $\tilde{G}_{m, \ell}$ ) of all  $m$ -subspaces (respectively oriented  $m$ -subspaces) of  $\mathbb{R}^{\ell+m}$ . Let  $\gamma_{G_m}^m$  and  $\hat{\gamma}_{G_m}^\ell$  denote the canonical vector bundles of dimensions  $m$  and  $\ell$  over the space  $G_m$  respectively such that  $\gamma_{G_m}^m \oplus \hat{\gamma}_{G_m}^\ell$  is the trivial bundle  $\varepsilon_{G_m}^{\ell+m}$ . Let  $T(\hat{\gamma}_{G_m}^\ell)$  denote the Thom space of  $\hat{\gamma}_{G_m}^\ell$ . The spaces  $\{T(\hat{\gamma}_{G_m}^\ell)\}_\ell$  constitute a spectrum. Let  $i^G : G_n \rightarrow G_{n+1}$  denote the injection mapping an  $n$ -plane  $a$  to the  $(n + 1)$ -plane generated by  $a$  and the  $(n + \ell + 1)$ -th unit vector  $\mathbf{e}_{n+\ell+1}$  in  $\mathbb{R}^{n+\ell+1}$ . Let  $\Omega = \Omega(\gamma_{G_n}^n, TP)$  and  $\Omega_\star = \Omega_\star(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1)$  be the open subbundles of  $J^k(\gamma_{G_n}^n, TP)$  and  $J^k(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1)$  associated to  $\Omega(n, p)$  and  $\Omega_\star(n + 1, p + 1)$  respectively, where  $\varepsilon_P^1$  denotes the trivial bundle  $P \times \mathbb{R}$ . Set  $\hat{\gamma}_\Omega^\ell = (\pi_{G_n}^k)^*(\hat{\gamma}_{G_n}^\ell)|_\Omega$  and  $\hat{\gamma}_{\Omega_\star}^\ell = (\pi_{G_{n+1}}^k)^*(\hat{\gamma}_{G_{n+1}}^\ell)|_{\Omega_\star}$ . There exists a fiberwise map  $\Delta^{(\Omega, \Omega_\star)} : \Omega \rightarrow \Omega_\star$  associated to  $i_{+1}|\Omega(n, p) : \Omega(n, p) \rightarrow \Omega_\star(n + 1, p + 1)$  covering  $i^G \times id_P$ . Then  $\Delta^{(\Omega, \Omega_\star)}$  induces the bundle map

$$\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_\star)} : \hat{\gamma}_\Omega^\ell \longrightarrow \hat{\gamma}_{\Omega_\star}^\ell$$

covering  $\Delta^{(\Omega, \Omega_\star)}$  and the associated map  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_\star)})$  between the Thom spaces. Let  $\ell \gg n, p$ . We denote the image of

$$(1.2) \quad T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_\star)})_* : \lim_{\ell \rightarrow \infty} \pi_{n+\ell}(T(\hat{\gamma}_\Omega^\ell)) \longrightarrow \lim_{\ell \rightarrow \infty} \pi_{n+\ell}(T(\hat{\gamma}_{\Omega_\star}^\ell))$$

by  $\text{Im}^{\mathfrak{N}}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$  (respectively  $\text{Im}^{\mathfrak{D}}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$ ) in the unoriented (respectively the oriented) case.

We are ready to state the main result of this paper.

**Theorem 1.1.** *Let  $n$  and  $p$  be positive integers. Let  $P$  be a  $p$ -dimensional manifold. Let  $\Omega(n, p)$  and  $\Omega_*(n+1, p+1)$  denote the above open subsets invariant with respect to the actions of  $L^k(p) \times L^k(n)$  and  $L^k(p+1) \times L^k(n+1)$ , respectively, such that  $i_{+1}(\Omega(n, p)) \subset \Omega_*(n+1, p+1)$ ,  $\Omega(n, p)$  satisfies the  $h$ -principle and that  $\Omega_*(n+1, p+1)$  satisfies the relative  $h$ -principle on the existence level in  $(h-P)$ .*

*Then there exist isomorphisms*

$$\begin{aligned} \mathfrak{n}^{(\Omega, \Omega_*)} : \mathfrak{N}(n, P; \Omega, \Omega_*) &\longrightarrow \text{Im}^{\mathfrak{N}}\left(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)})\right), \\ \mathfrak{o}^{(\Omega, \Omega_*)} : \mathfrak{D}(n, P; \Omega, \Omega_*) &\longrightarrow \text{Im}^{\mathfrak{D}}\left(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)})\right). \end{aligned}$$

Let  $\mathcal{K}$  denote the contact group introduced in [36] which acts on  $J^k(n, p)$  or  $J^k(n+m, p+m)$ . Let  $\Omega_{\mathcal{K}}^m = \Omega_{\mathcal{K}}(n+m, p+m)$  denote the subset of  $J^k(n+m, p+m)$  which consists of all  $\mathcal{K}$ -orbits  $\mathcal{K}(i_{+m}(z))$  for  $k$ -jets  $z \in \Omega(n, p)$ . It will be proved that  $\Omega_{\mathcal{K}}(n+m, p+m)$  is an open subset (see Lemma 5.2).

As for the image of  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)})_*$ , we will prove the following theorem.

**Theorem 1.2.** *Let  $n < p$  and  $P$  be as in Theorem 1.1. Let  $\Omega(n, p)$  denote a nonempty open subset in  $J^k(n, p)$  invariant with respect to the action of  $\mathcal{K}$  and let  $\Omega_{\mathcal{K}}(n+1, p+1)$  be as above. Then the homomorphism  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_{\mathcal{K}})})_*$  in (1.2) is surjective.*

Let  $\Omega_{\mathcal{K}}(\gamma_{\tilde{G}_{n+m, \ell}}^{n+m}, \mathbb{R}^{p+m})$  denote the subbundle of  $J^k(\gamma_{\tilde{G}_{n+m, \ell}}^{n+m}, \mathbb{R}^{p+m})$  associated to  $\Omega_{\mathcal{K}}(n+m, p+m)$ . For  $\mathfrak{D} = \mathfrak{D}(n, P; \Omega, \Omega_{\mathcal{K}})$  we define

$$B_{\mathfrak{D}} = \lim_{m \rightarrow \infty} \left( \lim_{\ell \rightarrow \infty} C^0(S^{\ell+n-p}, T((\pi_{\tilde{G}_{n+m, \ell}}^k)^* \hat{\gamma}_{\tilde{G}_{n+m, \ell}}^{\ell} |_{\Omega_{\mathcal{K}}(\gamma_{\tilde{G}_{n+m, \ell}}^{n+m}, \mathbb{R}^{p+m}))) \right),$$

where  $C^0(X, Y)$  denotes the space consisting of all base-point preserving continuous maps between connected spaces with base points equipped with the compact-open topology.

**Theorem 1.3.** *Let  $n < p$  or  $n \geq p \geq 2$ . Let  $P$  be a closed connected oriented  $p$ -dimensional manifold. Let  $\Omega(n, p)$  be a nonempty  $\mathcal{K}$ -invariant open subset such that if  $n \geq p \geq 2$ , then  $\Omega(n, p)$  contains all fold jets at least. Then there exists an isomorphism*

$$\mathfrak{D}(n, P; \Omega, \Omega_{\mathcal{K}}) \longrightarrow [P, B_{\mathfrak{D}}].$$

In our study of cobordisms of singular maps the  $h$ -principle for  $\Omega$ -regular maps plays a quite important role and our method is available in the dimensions  $n \geq p$  as well. Theorem 1.1 has been developed from an observation for fold-maps in [4, Theorem 0.4]. Set  $\Omega_{\mathcal{K}}^m = \Omega_{\mathcal{K}}(\gamma_{G_{n+m}}^{n+m}, TP \oplus \varepsilon_P^m)$ . Theorem 1.2 implies that  $\mathfrak{D}(n, P; \Omega, \Omega_{\mathcal{K}})$  is isomorphic to the stable homotopy group  $\lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^{\ell}}^{\ell}) \right)$ . If we apply the relative  $h$ -principle in Theorem 9.1 to [42, Theorem 9.2] due to Sadykov, then it follows that  $\mathfrak{D}(n, P; \Omega, \Omega_{\mathcal{K}})$  is isomorphic to  $\pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^m}^{\ell}) \right)$  also in the dimensions  $n \geq p \geq 2$  as well as  $n < p$ , where  $\hat{\gamma}_{\Omega_{\mathcal{K}}^m}^{\ell} = (\pi_{G_{n+m}}^k)^* (\hat{\gamma}_{G_{n+m}}^{\ell}) |_{\Omega_{\mathcal{K}}^m}$  and  $\ell$  and  $m$  are sufficiently large integers. By using these stable homotopy groups,

we will induce the space  $B_{\mathfrak{D}}$  by applying the  $S$ -duality in [47] in Section 8. In Section 9 we will see that the h-principle in (h-P) holds for a very wide class of  $\mathcal{K}$ -invariant open sets  $\Omega(n, p)$ . Let  $\Sigma^I(n, p)$  denote the Boardman manifold in  $J^k(n, p)$  with Boardman symbol  $I = (i_1, \dots, i_k)$  defined in [12]. An important example of  $\Omega(n, p)$  is the open subset  $\Omega^I(n, p)$  which is the union of all  $\Sigma^K(n, p)$  with  $K \leq I$  in the lexicographic order. The results of the paper show the importance of the homotopy type of  $\Omega(n, p)$ . In [1], [2], [4] and [5] we have described the homotopy type of  $\Omega^{i,0}(n, p)$  for  $i = \max\{n - p + 1, 1\}$  in terms of orthogonal groups and Stiefel manifolds. Although available only in the case of fold-maps, we will construct a simpler spectrum in Section 10 whose stable homotopy group is a direct summand of  $\mathfrak{D}(n, P; \Omega^{i,0}, \Omega^{i,0})$  (respectively  $\mathfrak{N}(n, P; \Omega^{i,0}, \Omega^{i,0})$ ) and also construct the corresponding classifying space. This result is a refinement of [4, Theorem 0.3]. Let  $F_m$  denote the space of all base point preserving maps of the  $m$ -sphere  $S^m$  and let  $F = \lim_{m \rightarrow \infty} F_m$ . When  $n = p \geq 1$  and  $P$  is closed and oriented, we will prove that there exists an isomorphism of  $\mathfrak{D}(n, P; \Omega^{1,0}, \Omega^{1,0})$  onto  $[P, F]$ , which has been given in [3] and [4] from a different point of view. Therefore, we may assert that the topology of  $B_{\mathfrak{D}}$  will be important in connection with the canonical homomorphism of  $\mathfrak{D}(n, S^n; \Omega^{1,0}, \Omega^{1,0})$  to  $\mathfrak{D}(n, S^n; \Omega, \Omega_{\mathcal{K}})$  and the map  $F \rightarrow B_{\mathfrak{D}}$ . We should refer to Chess[14], in which  $\mathfrak{D}(n, \mathbb{R}^n; \Omega^{1,0}, \Omega^{1,0}) \approx \pi_n^s$  has been proved.

The study of h-principles for  $\Omega$ -regular maps has a long history. Here, we only refer the reader to Smale[46], Hirsch[25], Phillips[39], Feit[22], Gromov([23], [24]), Eliashberg([20], [21]) and du Plessis([16], [17], [18]) for the details and further references.

In [19] Eliashberg has studied the cobordisms of the solutions of the first order differential relations such as Lagrange and Legendre immersions by applying his h-principle. Sadykov[42] has studied and expressed the cobordism group of  $\Omega$ -regular maps in terms of the stable homotopy group as explained above under the assumption of the relative h-principle in a formal approach.

In [40] Rimányi and Szűcs have constructed a certain classifying space such that the group of the cobordism classes of smooth maps of  $n$ -dimensional manifolds into  $P$  having only a given class of  $C^\infty$  stable singularities is described by the homotopy classes of  $P$  to this space in the case  $n < p$  and in [51] Szűcs has developed further results. The method of the construction of this classifying space is quite different from ours of the space  $B_{\mathfrak{D}}$  using h-principles in this paper. In [28], [29] and [30] Kalmár has studied the cobordism groups of fold-maps in negative codimensions.

As another line of investigation of cobordisms of singular maps in which h-principles are not available, we refer to Saeki[43], Ikegami-Saeki[26], Saeki[44], Kalmár[27] and Sadykov[41].

We will define the homomorphisms  $\mathfrak{n}^{(\Omega, \Omega_*)}$  and  $\mathfrak{o}^{(\Omega, \Omega_*)}$  in Section 3. In Section 4 we will prove Theorem 1.1. In Section 5 we will prove that  $\Omega_{\mathcal{K}}^m$  is an open subset. In Section 6 we will prepare several results which are necessary in the proofs of Theorems 1.2 and 1.3. In Section 7 we will prove Theorem 1.2. In Section 8 we will explain how the space  $B_{\mathfrak{D}}$  is introduced. In Section 9 we will give a wide class of open sets  $\Omega(n, p)$  which satisfy the h-principle in (h-P). In Section 10 we will construct a simpler spectrum and then, the classifying space  $B_V$  such that  $[P, B_V]$  is a direct summand of the cobordism group for fold-maps by using Theorem 1.1 and the homotopy type of  $\Omega^{n-p+1,0}(n, p)$ .

## 2. PRELIMINARIES

Given a fiber bundle  $\pi^\mathcal{E} : \mathcal{E} \rightarrow X$  and a subset  $C$  in  $X$ , we denote  $(\pi^\mathcal{E})^{-1}(C)$  by  $\mathcal{E}|_C$ . Let  $\pi^\mathcal{F} : \mathcal{F} \rightarrow Y$  be another fiber bundle. A map  $\tilde{b} : \mathcal{E} \rightarrow \mathcal{F}$  is called a fiber map covering a map  $b : X \rightarrow Y$  if  $\pi^\mathcal{F} \circ \tilde{b} = b \circ \pi^\mathcal{E}$  holds. The restriction  $\tilde{b}|(\mathcal{E}|_C) : \mathcal{E}|_C \rightarrow \mathcal{F}$  (or  $\mathcal{F}|_{b(C)}$ ) is denoted by  $\tilde{b}|_C$ . In particular, for a point  $x \in X$ ,  $\mathcal{E}|_x$  and  $\tilde{b}|_x$  are simply denoted by  $\mathcal{E}_x$  and  $\tilde{b}_x : \mathcal{E}_x \rightarrow \mathcal{F}_{b(x)}$  respectively. The trivial bundle  $X \times \mathbb{R}^\ell$  is denoted by  $\varepsilon_X^\ell$  ([49]).

Let  $\mathcal{E} \rightarrow X$  and  $\mathcal{F} \rightarrow Y$  be vector bundles of dimensions  $n$  and  $p$  respectively. The origin of  $\mathbb{R}^m$  is simply denoted by 0 for any natural number  $m$ . We define the action of  $L^k(p) \times L^k(n)$  on  $J^k(n, p)$  by  $(j_0^k h_1, j_0^k h_2) \cdot j_0^k f = j_0^k (h_1 \circ f \circ h_2^{-1})$ . Let  $\mathcal{E}_1 \rightarrow X_1$  and  $\mathcal{F}_1 \rightarrow Y_1$  be other vector bundles of dimensions  $n$  and  $p$  respectively, and let  $\tilde{b}_1 : \mathcal{E} \rightarrow \mathcal{E}_1$  and  $\tilde{b}_2 : \mathcal{F} \rightarrow \mathcal{F}_1$  be bundle maps covering  $b_1 : X \rightarrow X_1$  and  $b_2 : Y \rightarrow Y_1$  respectively. Then  $\tilde{b}_1$  and  $\tilde{b}_2$  yield the isomorphisms  $S^i(\mathcal{E}_x) \rightarrow S^i(\mathcal{E}_{1, b_1(x)})$  and  $\mathcal{F}_y \rightarrow \mathcal{F}_{1, b_2(y)}$  for any  $x \in X$  and  $y \in Y$  for  $1 \leq i \leq k$  and hence, we have the bundle map

$$(2.1) \quad \mathbf{j}(\tilde{b}_1, \tilde{b}_2) : J^k(\mathcal{E}, \mathcal{F}) \rightarrow J^k(\mathcal{E}_1, \mathcal{F}_1)$$

covering  $b_1 \times b_2$ . Then  $\mathbf{j}(\tilde{b}_1, \tilde{b}_2)$  induces the bundle map  $\mathbf{j}(\tilde{b}_1, \tilde{b}_2)_\Omega : \Omega(\mathcal{E}, \mathcal{F}) \rightarrow \Omega(\mathcal{E}_1, \mathcal{F}_1)$ .

If we provide  $N$  and  $P$  with Riemannian metrics, then the Levi-Civita connections induce the exponential maps  $\exp_{N,x} : T_x N \rightarrow N$  and  $\exp_{P,y} : T_y P \rightarrow P$  for  $x \in N$  and  $y \in P$  respectively. In dealing with the exponential maps we always consider the convex neighborhoods ([33]). We define the smooth bundle map

$$(2.2) \quad J^k(N, P) \rightarrow J^k(TN, TP) \quad \text{over } N \times P$$

by sending  $z = j_x^k f \in (\pi_N^k \times \pi_P^k)^{-1}(x, y)$  to the  $k$ -jet of  $(\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}$  at  $\mathbf{0} \in T_x N$ , which is regarded as an element of  $J^k(T_x N, T_y P) (= J_{x,y}^k(TN, TP))$  (see [33, Proposition 8.1] for the smoothness of exponential maps). More strictly, (2.2) gives a smooth equivalence of the fiber bundles under the structure group  $L^k(p) \times L^k(n)$ . Namely, it gives a smooth reduction of the structure group  $L^k(p) \times L^k(n)$  of  $J^k(N, P)$  to  $O(p) \times O(n)$ , which is the structure group of  $J^k(TN, TP)$ . Let  $\Sigma^I(N, P)$  denote the Boardman manifold in  $J^k(N, P)$ , where  $I = (i_1, \dots, i_k)$  is a Boardman symbol such that  $n \geq i_1 \geq \dots \geq i_k \geq 0$  (see [12], [34] and [38]). Let  $\Omega^I(N, P)$  denote the open subset which is the union of all  $\Sigma^K(N, P)$  with  $K \leq I$  in the lexicographic order. Let  $\Sigma^I(TN, TP)$  and  $\Omega^I(TN, TP)$  be the subbundles of  $J^k(TN, TP)$  associated to  $\Sigma^I(n, p)$  and  $\Omega^I(n, p)$ , which are identified with  $\Sigma^I(N, P)$  and  $\Omega^I(N, P)$  under (2.2), respectively.

Let  $f : X \rightarrow Y$  be a continuous map. If  $f_* : \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $0 \leq i < m$  and an epimorphism for  $i = m$ , then we call  $f$  a homotopy  $m$ -equivalence in this paper.

3. HOMOMORPHISMS  $\mathfrak{n}^{(\Omega, \Omega_*)}$  AND  $\mathfrak{o}^{(\Omega, \Omega_*)}$ 

Let  $\Omega(n, p)$  and  $\Omega_*(n+1, p+1)$  be open subsets given in Theorem 1.1.

As usual we provide  $\mathfrak{N}(n, P; \Omega, \Omega_*)$  and  $\mathfrak{D}(n, P; \Omega, \Omega_*)$  with the structures of monoids. Namely, given two  $\Omega$ -regular maps  $f_i : N_i \rightarrow P$  ( $i = 0, 1$ ), we define the sum  $[f_0] + [f_1]$  to be the cobordism class of the  $\Omega$ -regular map  $f : N_0 \cup N_1 \rightarrow P$  defined by  $f|N_i = f_i$ . We proceed the arguments commonly in the unoriented case and the oriented case in Sections 3 and 4, because in the oriented case we only

need to provide manifolds and vector bundles concerned with the orientations. The orientability of  $P$  is not necessary even in the oriented case in Theorems 1.1 and 1.2. We will need the orientability of  $P$  at the end of Section 8.

We will identify  $\mathbb{R}^m$  with  $S^m \setminus \{(0, \dots, 0, 1)\}$  for positive integers  $m$  in the following. Let  $M$  be an  $m$ -dimensional compact manifold such that  $M$  should be oriented in the oriented case. Let  $\ell \gg m$ . Take an embedding  $e_M : M \rightarrow \mathbb{R}^{\ell+m}$  and identify  $M$  with  $e_M(M)$ . Let  $c_M : M \rightarrow G_m$  be the classifying map defined by sending a point  $x \in M$  to the  $m$ -plane  $T_x M \in G_m$ . Let  $\nu_M$  be the orthogonal normal bundle of  $M$  in  $\mathbb{R}^{\ell+m}$ . Let  $c_{TM} : TM \rightarrow \gamma_{G_m}^m$  (respectively  $c_{\nu_M} : \nu_M \rightarrow \hat{\gamma}_{G_m}^\ell$ ) be the bundle map covering the classifying map  $c_M : M \rightarrow G_m$ , which is defined by sending a vector  $\mathbf{v}$  of  $T_x M$  (respectively  $\mathbf{w} \in \nu_{M,x}$ ) to  $(T_x M, \mathbf{v})$  (respectively  $(T_x M, \mathbf{w})$ ). Then we have the canonical trivializations  $t_M : TM \oplus \nu_M \rightarrow \varepsilon_M^{\ell+m}$  and  $t_{G_m} : \gamma_{G_m}^m \oplus \hat{\gamma}_{G_m}^\ell \rightarrow \varepsilon_{G_m}^{\ell+m}$ . They induce  $t_{G_m} \circ (c_{TM} \oplus c_{\nu_M}) \circ t_M^{-1} = c_M \times id_{\mathbb{R}^{\ell+m}}$ . Let  $\mathcal{F}$  be any vector bundle of dimension  $q$  over  $P$  and  $\Omega(m, q)$  be an  $L^k(q) \times L^k(m)$ -invariant open subspace of  $J^k(m, q)$ . Let  $\Omega_m = \Omega(\gamma_{G_m}^m, \mathcal{F})$  be the subbundle of  $J^k(\gamma_{G_m}^m, \mathcal{F})$  associated to  $\Omega(m, q)$ . If there is a map  $s_M : M \rightarrow \Omega_m$  with  $\pi_{G_m}^k \circ s_M$  being homotopic (respectively equal) to  $c_M$ , then  $c_{TM}$ ,  $c_{\nu_M}$  and the projection  $\pi_{G_m}^k|_{\Omega_m} : \Omega_m \rightarrow G_m$  induce the bundle maps  $c_{s_M}^{TM} : TM \rightarrow (\pi_{G_m}^k)^* \gamma_{G_m}^m|_{\Omega_m}$  and  $c_{s_M}^{\nu_M} : \nu_M \rightarrow (\pi_{G_m}^k)^* \hat{\gamma}_{G_m}^\ell|_{\Omega_m}$  covering  $s_M$  such that  $t_{\Omega_m} \circ (c_{s_M}^{TM} \oplus c_{s_M}^{\nu_M}) \circ t_M^{-1}$  is homotopic (respectively equal) to  $s_M \times id_{\mathbb{R}^{\ell+m}}$ , where  $t_{\Omega_m} : (\pi_{G_m}^k)^* (\gamma_{G_m}^m \oplus \hat{\gamma}_{G_m}^\ell)|_{\Omega_m} \rightarrow \varepsilon_{\Omega_m}^{\ell+m}$  is the trivialization induced from  $t_{G_m}$ .

$$\begin{array}{ccc}
 & \Omega(\gamma_{G_m}^m, \mathcal{F}) & \longleftarrow (\pi_{G_m}^k)^* \gamma_{G_m}^m|_{\Omega_m} \text{ (or } (\pi_{G_m}^k)^* \hat{\gamma}_{G_m}^\ell|_{\Omega_m}) \\
 M \nearrow s_M & \downarrow & \uparrow c_{s_M}^{TM} \text{ (or } c_{s_M}^{\nu_M}) \\
 & G_m \times P & TM \text{ (or } \nu_M) \\
 M \searrow c_M & \downarrow & \downarrow c_{TM} \text{ (or } c_{\nu_M}) \\
 & G_m & \longleftarrow \gamma_{G_m}^m \text{ (or } \hat{\gamma}_{G_m}^\ell)
 \end{array}$$

Let  $A_{m,q}$  refer to the grassmann manifold  $G_{m,q}$  (respectively oriented grassmann manifold  $\tilde{G}_{m,q}$ ) of all  $m$ -subspaces (respectively oriented  $m$ -subspaces) of  $\mathbb{R}^{m+q}$ . Let  $\gamma_{A_{m,q}}^m$  and  $\hat{\gamma}_{A_{m,q}}^q$  denote the canonical vector bundles of dimensions  $m$  and  $q$  over  $A_{m,q}$  respectively such that  $\gamma_{A_{m,q}}^m \oplus \hat{\gamma}_{A_{m,q}}^q$  is the trivial bundle  $\varepsilon_{A_{m,q}}^{m+q}$ . Let  $T(\hat{\gamma}_{A_{m,q}}^q)$  denote the Thom space of  $\hat{\gamma}_{A_{m,q}}^q$ . Here, we see that the spaces  $\{T((\pi_{G_m}^k)^* \hat{\gamma}_{G_m}^\ell|_{\Omega_m})\}_\ell$  constitute a spectrum. Let  $j : G_m \rightarrow A_{m,\ell+1}$  denote the injection mapping an  $m$ -plane  $a$  in  $\mathbb{R}^{m+\ell} = \mathbb{R}^{m+\ell} \times 0$  to the same  $a$  as the  $m$ -plane in  $\mathbb{R}^{m+\ell+1}$ . We have the canonical bundle maps

$$b : \gamma_{G_m}^m \longrightarrow \gamma_{A_{m,\ell+1}}^m \quad \text{and} \quad \hat{b} : \hat{\gamma}_{G_m}^\ell \oplus \varepsilon_{G_m}^1 \longrightarrow \hat{\gamma}_{A_{m,\ell+1}}^{\ell+1}$$

covering  $j$ . Let  $\Omega_m^\bullet = \Omega(\gamma_{A_{m,\ell+1}}^m, \mathcal{F})$  be the open subbundle of  $J^k(\gamma_{A_{m,\ell+1}}^m, \mathcal{F})$  associated to  $\Omega(m, q)$ . We have the bundle map

$$\mathbf{j}(b, id_{\mathcal{F}})_\Omega : \Omega_m \longrightarrow \Omega_m^\bullet$$

covering  $j$ . Then  $\hat{b}$  induces the bundle map

$$\mathbf{b} : (\pi_{G_m}^k)^* (\hat{\gamma}_{G_m}^\ell \oplus \varepsilon_{G_m}^1)|_{\Omega_m} \longrightarrow (\pi_{A_{m,\ell+1}}^k)^* (\hat{\gamma}_{A_{m,\ell+1}}^{\ell+1})|_{\Omega_m^\bullet}$$

covering  $\mathbf{j}(b, id_{\mathcal{F}})_\Omega$ . Since  $T((\pi_{G_m}^k)^* (\hat{\gamma}_{G_m}^\ell \oplus \varepsilon_{G_m}^1)|_{\Omega_m}) = T((\pi_{G_m}^k)^* \hat{\gamma}_{G_m}^\ell|_{\Omega_m}) \wedge S^1$ , we have the associated map

$$T(\mathbf{b}) : T((\pi_{G_m}^k)^* (\hat{\gamma}_{G_m}^\ell)|_{\Omega_m}) \wedge S^1 \longrightarrow T((\pi_{A_{m,\ell+1}}^k)^* (\hat{\gamma}_{A_{m,\ell+1}}^{\ell+1})|_{\Omega_m^\bullet}).$$

This shows the assertion.

Take an embedding  $e_N : N \rightarrow \mathbb{R}^{n+\ell} \subset S^{n+\ell}$  and apply the above notation. Then we have the bundle map

$$\mathbf{j}(c_{TN}, id_{TP})_\Omega : \Omega(TN, TP) \rightarrow \Omega = \Omega(\gamma_{G_n}^n, TP).$$

Let  $f : N \rightarrow P$  be an  $\Omega$ -regular map with the jet extension  $j^k f : N \rightarrow \Omega(TN, TP)$  and let  $s_N = \mathbf{j}(c_{TN}, id_{TP})_\Omega \circ j^k f$ . Then we have the composite

$$\Delta^{(\Omega, \Omega_*)} \circ s_N : N \rightarrow \Omega_* = \Omega_*(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1)$$

and

$$\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)} \circ c_{s_N}^{\nu_N} : \nu_N \rightarrow \hat{\gamma}_{\Omega_*}^\ell$$

covering  $\Delta^{(\Omega, \Omega_*)} \circ s_N$ . We denote the Pontrjagin-Thom construction for  $e_N$  by  $a_N : S^{n+\ell} \rightarrow T(\nu_N)$  ([52]). Let  $\mathfrak{C}(n, P; \Omega, \Omega_*)$  refer to  $\mathfrak{N}(n, P; \Omega, \Omega_*)$  or  $\mathfrak{D}(n, P; \Omega, \Omega_*)$ , let  $\text{Im}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$  refer to  $\text{Im}^{\mathfrak{N}}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$  or  $\text{Im}^{\mathfrak{D}}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$  and let  $\omega$  refer to  $\mathfrak{n}^{(\Omega, \Omega_*)}$  or  $\mathfrak{o}^{(\Omega, \Omega_*)}$ , depending on whether we work in the unoriented case or oriented case. We now define the maps

$$\omega : \mathfrak{C}(n, P; \Omega, \Omega_*) \longrightarrow \text{Im}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$$

by mapping the cobordism class  $[f]$  of  $\mathfrak{C}(n, P; \Omega, \Omega_*)$  to the homotopy class of  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{s_N}^{\nu_N}) \circ a_N$ .

We have to prove that  $\omega([f])$  does not depend on the choice of a representative  $f$ .

**Lemma 3.1.** *Suppose that two  $\Omega$ -regular maps  $f_i : N_i \rightarrow P$  ( $i = 0, 1$ ) are  $\Omega_*$ -cobordant. Then we have  $\mathfrak{n}^{(\Omega, \Omega_*)}([f_0]) = \mathfrak{n}^{(\Omega, \Omega_*)}([f_1])$ . If  $N_i$  are oriented and  $f_i$  ( $i = 0, 1$ ) are oriented  $\Omega_*$ -cobordant, then we have  $\mathfrak{o}^{(\Omega, \Omega_*)}([f_0]) = \mathfrak{o}^{(\Omega, \Omega_*)}([f_1])$ .*

*Proof.* We first have to prove that  $\omega([f])$  does not depend on the choice of an embedding  $e_N$ . Let  $\ell$  be a sufficiently large integer. Let  $e_N : N \rightarrow \mathbb{R}^{n+\ell}$  be an embedding of  $N$ . For a nonnegative integer  $m$ , let  $e_N^m$  be the composite of  $e_N$  and the inclusion  $\mathbb{R}^{n+\ell} = \mathbb{R}^{n+\ell} \times 0 \rightarrow \mathbb{R}^{n+\ell+m}$ . Then we may regard  $\nu_N \oplus \varepsilon_N^m$  as the normal bundle of  $e_N^m$  and the Pontrjagin-Thom construction for  $e_N^m$  is the  $m$ -th suspension  $S^{n+\ell+m} \rightarrow T(\nu_N \oplus \varepsilon_N^m) = T(\nu_N) \wedge S^m$  of that for  $e_N$ .

Given two embeddings  $e_{N_i} : N_i \rightarrow \mathbb{R}^{n+\ell_i}$  ( $i = 0, 1$ ) with  $\ell_1 = \ell_0 + m$  for  $m \geq 0$ , by the above argument we consider  $e_{N_0}^m$  in place of  $e_{N_0}$ . It will be proved by the argument below that  $\omega([f])$  does not depend on the choice of  $e_{N_0}^m$  and  $e_{N_1}$ .

Let  $\epsilon$  be a sufficiently small positive real number. Let  $I(0, \epsilon)$  and  $I(1, \epsilon)$  denote the intervals  $[0, \epsilon]$  and  $[1 - \epsilon, 1]$  respectively. Let  $\mathfrak{C} : (W, \partial W) \rightarrow (P \times [0, 1], P \times 0 \cup P \times 1)$  be an  $\Omega_*$ -cobordism of  $f_0$  and  $f_1$ . Take embeddings  $e_{N_i} : N_i \rightarrow \mathbb{R}^{n+\ell}$  and  $e_W : W \rightarrow \mathbb{R}^{n+\ell} \times [0, 1]$ , and let us identify as  $N_i = e_{N_i}(N_i) = e_{N_i}(N_i) \times \{i\}$ ,  $W = e_W(W)$  and  $P = P \times \{i\}$ . Then we may assume that for  $i = 0, 1$ ,

- (i)  $W \cap (S^{n+\ell} \times I(i, \epsilon)) = N_i \times I(i, \epsilon)$ ,
- (ii)  $e_W|_{N_i \times I(i, \epsilon)} = e_{N_i} \times id_{I(i, \epsilon)}$ ,
- (iii)  $\mathfrak{C}|_{N_i \times I(i, \epsilon)} = f_i \times id_{I(i, \epsilon)}$ ,
- (iv)  $j^k \mathfrak{C}|_{N_i \times \{i\}} = j^k(f_i \times id_{\mathbb{R}})|_{N_i \times \{i\}}$  under  $N_i = N_i \times \{i\}$  and  $P = P \times \{i\}$ .

In the identification  $TW|_{N_i} = TN_i \oplus \varepsilon_{N_i}^1$ , the positive direction of  $\varepsilon_{N_0}^1$  should correspond to the inward normal direction, and that of  $\varepsilon_{N_1}^1$  should correspond to the outward normal direction. Then we may assume that the trivializations

$$t_{N_i} : TN_i \oplus \nu_{N_i} \rightarrow \varepsilon_{N_i}^{n+\ell} \quad \text{and} \quad t_W : TW \oplus \nu_W \rightarrow \varepsilon_W^{n+\ell+1}$$

satisfy  $t_W|_{N_i} = (t_{N_i} \oplus id_{\varepsilon_{N_i}^1}) \circ (id_{TN_i} \oplus \mathbf{k}_{N_i}^\sim)$ , where  $\mathbf{k}_{N_i}^\sim : \varepsilon_{N_i}^1 \oplus \nu_{N_i} \rightarrow \nu_{N_i} \oplus \varepsilon_{N_i}^1$  is the map interchanging the components  $\varepsilon_{N_i}^1$  and  $\nu_{N_i}$ . Let  $c_{\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1} : \gamma_{G_n}^n \oplus \varepsilon_{G_n}^1 \rightarrow \gamma_{G_{n+1}}^{n+1}$  denote the bundle map covering  $i^G : G_n \rightarrow G_{n+1}$  defined by  $c_{\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1}(V_x \oplus (x, t)) = V_{i^G(x)} + t\mathbf{e}_{n+\ell+1}$  for  $x \in G_n$ ,  $V_x \in (\gamma_{G_n}^n)_x$  and  $t \in \mathbb{R}$ . Let  $s_{N_i} = \mathbf{j}(c_{TN_i}, id_{TP})_\Omega \circ j^k f_i$  and  $s_W = \mathbf{j}(c_{TW}, id_{T(P \times [0,1])})_{\Omega_*} \circ j^k \mathfrak{E}$ . Then we have that

$$\begin{aligned} c_{\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1} \circ (c_{TN_i} \oplus (c_{N_i} \times id_{\mathbb{R}})) &= c_{TW}|_{TN_i} \oplus \varepsilon_{N_i}^1, \\ c_{s_W}^{\nu_W}|_{N_i} &= \mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)} \circ c_{s_{N_i}}^{\nu_{N_i}}. \end{aligned}$$

Let  $a_W : S^{n+\ell} \times [0, 1] \rightarrow T(\nu_W)$  be the Pontrjagin-Thom construction for  $e_W$ . Under the identifications

$$\begin{aligned} \Omega_*(TW, T(P \times [0, 1])) &= \Omega_*(TW, TP \oplus \varepsilon_P^1) \times [0, 1], \\ \Omega_*(\gamma_{G_{n+1}}^{n+1}, T(P \times [0, 1])) &= \Omega_*(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1) \times [0, 1], \end{aligned}$$

the composite  $T(c_{s_W}^{\nu_W}) \circ a_W$  gives a homotopy between  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{s_{N_0}}^{\nu_{N_0}}) \circ a_{N_0}$  and  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{s_{N_1}}^{\nu_{N_1}}) \circ a_{N_1}$ . This proves  $\omega([f_0]) = \omega([f_1])$ .  $\square$

#### 4. $\mathfrak{n}^{(\Omega, \Omega_*)}$ AND $\mathfrak{o}^{(\Omega, \Omega_*)}$ ARE ISOMORPHISMS

We prove Theorem 1.1 in this section. Let  $\Omega(n, p)$  and  $\Omega_*(n+1, p+1)$  be the open subsets given in Theorem 1.1.

*Proof of Theorem 1.1.* We use the notation in the proof of Lemma 3.1. We first prove that  $\omega$  is injective. For this, take two  $\Omega$ -regular maps  $f_i : N_i \rightarrow P$  ( $i = 0, 1$ ) such that  $\omega([f_0]) = \omega([f_1])$ . Recall the map  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{s_{N_i}}^{\nu_{N_i}}) \circ a_{N_i}$  which represents  $\omega([f_i])$ . There is a homotopy  $H : S^{n+\ell} \times [0, 1] \rightarrow T(\hat{\gamma}_{\Omega_*}^\ell)$  such that if  $\epsilon$  is sufficiently small, then we have, for  $i = 0, 1$ ,

- (i)  $H|_{S^{n+\ell} \times I(i, \epsilon)} = T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{s_{N_i}}^{\nu_{N_i}}) \circ a_{N_i} \circ (\pi_{S^{n+\ell}}|_{S^{n+\ell} \times I(i, \epsilon)}),$
- (ii)  $H$  is smooth around  $H^{-1}(\Omega_*)$  and is transverse to  $\Omega_*$ .

We set  $W = H^{-1}(\Omega_*)$ . Then we have

- (iii)  $W \cap (S^{n+\ell} \times I(i, \epsilon)) = N_i \times I(i, \epsilon),$
- (iv)  $H|_{N_i \times I(i, \epsilon)} = \Delta^{(\Omega, \Omega_*)} \circ s_{N_i} \circ (\pi_{N_i}|_{N_i \times I(i, \epsilon)}),$
- (v)  $TW|_{N_i \times I(i, \epsilon)} = T(N_i \times I(i, \epsilon)) = (TN_i \oplus \varepsilon_{N_i}^1) \times I(i, \epsilon),$
- (vi)  $\nu_W|_{N_i \times I(i, \epsilon)} = \nu_{N_i} \times I(i, \epsilon).$

By (ii) we have the bundle map  $c_{\sim}^{\nu_W} : \nu_W \rightarrow \hat{\gamma}_{\Omega_*}^\ell$  covering  $H|_W : W \rightarrow \Omega_*$  such that

- (vii)  $c_{\sim}^{\nu_W}|_{N_i \times I(i, \epsilon)} = \mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)} \circ c_{s_{N_i}}^{\nu_{N_i}} \circ (\pi_{\nu_{N_i}}|_{\nu_{N_i} \times I(i, \epsilon)})$  by (i) and (iv).

By [2, Proposition 3.3] we obtain a bundle map of  $TW \oplus \varepsilon_W^3$  to  $(\pi_{G_{n+1}}^k)^*(\gamma_{G_{n+1}}^{n+1} \oplus \varepsilon_{G_{n+1}}^3)|_{\Omega_*}$  with a required property concerning the trivialization. By this property and the dimensional reason, we obtain a bundle map

$$c_{\sim}^{TW} : TW \rightarrow (\pi_{G_{n+1}}^k)^*(\gamma_{G_{n+1}}^{n+1})|_{\Omega_*}$$

covering  $H|_W : W \rightarrow \Omega_*$  induced from the above bundle map such that  $t_{\Omega_*} \circ (c_{\sim}^{TW} \oplus c_{\sim}^{\nu_W}) \circ t_W^{-1}$  is homotopic to  $(H|_W) \times id_{\mathbb{R}^{n+\ell+1}}$ . Since  $\gamma_{G_{n+1}}^{n+1}$  is the universal bundle ( $\ell \gg n$ ),  $c_{\sim}^{TW}$  is regarded as  $c_{H|_W}^{TW}$ . Let  $\mathbf{b}(\gamma \oplus \varepsilon^1)^{(\Omega, \Omega_*)} : (\pi_{G_n}^k)^*(\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1)|_{\Omega} \rightarrow (\pi_{G_{n+1}}^k)^*(\gamma_{G_{n+1}}^{n+1})|_{\Omega_*}$  covering  $\Delta^{(\Omega, \Omega_*)}$  be the bundle map induced from  $c_{\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1}^n$  in the proof of Lemma 3.1. Then we may assume by (iv), (v) and (vi) that



(viii)  $c_{\sim}^{TW}|_{(N_i \times I(i, \epsilon))}$  is equal to

$$\mathbf{b}(\gamma \oplus \varepsilon^1)^{(\Omega, \Omega_*)} \circ (c_{s_{N_i}}^{TN_i} \oplus (c_{s_{N_i}} \times id_{\mathbb{R}})) \circ \pi_{TN_i \oplus \varepsilon_{N_i}}|((TN_i \oplus \varepsilon_{N_i}) \times I(i, \epsilon)).$$

Hence,  $c_W$  is homotopic to  $\pi_{G_{n+1}}^k \circ H|W$  relative to  $(N_0 \times [0, \epsilon]) \cup (N_1 \times [1 - \epsilon, 1])$ . Let  $\pi_{TP \oplus \varepsilon_P^1} : T(P \times [0, 1]) = (TP \oplus \varepsilon_P^1) \times [0, 1] \rightarrow TP \oplus \varepsilon_P^1$  be the canonical bundle map covering the canonical projection  $\pi_P : P \times [0, 1] \rightarrow P$ . Then we have the bundle map

$$\begin{aligned} \mathbf{j}(c_{TW}, \pi_{TP \oplus \varepsilon_P^1})_{\Omega_*} : \Omega_*(TW, T(P \times [0, 1])) &= \Omega_*(TW, TP \oplus \varepsilon_P^1) \times [0, 1] \\ &\longrightarrow \Omega_* = \Omega_*(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1) \end{aligned}$$

covering  $c_W \times \pi_P$ . Therefore, since  $[0, 1]$  is contractible, there is a section  $s_W : W \rightarrow \Omega_*(TW, T(P \times [0, 1]))$  such that

$$\begin{aligned} \pi_{P \times [0, 1]}^k \circ s_W|_{N_i \times I(i, \epsilon)} &= f_i \times id_{I(i, \epsilon)}, \\ \pi_P \circ \pi_{P \times [0, 1]}^k \circ s_W &= \pi_P^k \circ (H|W), \end{aligned}$$

and that  $\mathbf{j}(c_{TW}, \pi_{TP \oplus \varepsilon_P^1})_{\Omega_*} \circ s_W$  is homotopic to  $H|W$  relative to  $(N_0 \times [0, \epsilon]) \cup (N_1 \times [1 - \epsilon, 1])$ .

Since  $\Omega_*(TW, T(P \times [0, 1]))$  satisfies the relative h-principle on the existence level, there exists an  $\Omega_*$ -regular map  $\mathfrak{C} : W \rightarrow P \times [0, 1]$  such that  $\mathfrak{C}(x, t) = f_0(x) \times t$  for  $0 \leq t \leq \epsilon$ ,  $\mathfrak{C}(x, t) = f_1(x) \times t$  for  $1 - \epsilon \leq t \leq 1$  and that  $j^k \mathfrak{C}$  is homotopic to  $s_W$  relative to  $(N_0 \times [0, \epsilon/2]) \cup (N_1 \times [1 - \epsilon/2, 1])$ . This implies that the  $\Omega$ -regular maps  $f_0$  and  $f_1$  are  $\Omega_*$ -cobordant. This proves that  $\omega$  is injective.

We next prove that  $\omega$  is surjective. Let an element  $\tilde{\alpha}$  of  $\text{Im}(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))$  be represented by a map  $\alpha : S^{n+\ell} \rightarrow T(\hat{\gamma}_{\Omega}^{\ell})$  such that  $(T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}))_*([\alpha]) = \tilde{\alpha}$ . We may suppose that  $\alpha$  is smooth around  $\alpha^{-1}(\Omega)$  and is transverse to  $\Omega$ . We set  $N = \alpha^{-1}(\Omega)$ . If  $N = \emptyset$ , then  $[\alpha]$  must be a null element, although we can deform  $\alpha$  so that  $N \neq \emptyset$  even in this case. Since  $\alpha$  is transverse to  $\Omega$ , we have the bundle map  $c_{\sim}^{\nu_N} : \nu_N \rightarrow \hat{\gamma}_{\Omega}^{\ell}$  covering  $\alpha|N$ . It follows from [2, Proposition 3.3] that there exists a bundle map

$$c_{\sim}^{TN \oplus \varepsilon_N^3} : TN \oplus \varepsilon_N^3 \rightarrow (\pi_{G_n}^k)^*(\gamma_{G_n}^n \oplus \varepsilon_{G_n}^3)|_{\Omega}$$

covering  $\alpha|N : N \rightarrow \Omega$  such that the composite

$$\begin{aligned} (t_{\Omega} \oplus id_{\varepsilon_{\Omega}^3}) \circ (id_{(\pi_{G_n}^k)^*(\gamma_{G_n}^n)} \oplus \mathbf{k}_{G_n}^{\sim}) \\ \circ (c_{\sim}^{TN \oplus \varepsilon_N^3} \oplus c_{\sim}^{\nu_N}) \circ (id_{TN} \oplus \mathbf{k}_N^{\sim}) \circ (t_N^{-1} \oplus id_{\varepsilon_N^3}) \end{aligned}$$

is homotopic to  $(\alpha|N) \times id_{\mathbb{R}^{n+\ell+3}}$ , where  $\mathbf{k}_{G_n}^{\sim} : \varepsilon_{G_n}^3 \oplus \hat{\gamma}_{G_n}^{\ell} \rightarrow \hat{\gamma}_{G_n}^{\ell} \oplus \varepsilon_{G_n}^3$  and  $\mathbf{k}_N^{\sim} : \nu_N \oplus \varepsilon_N^3 \rightarrow \varepsilon_N^3 \oplus \nu_N$  are the maps interchanging the components respectively. Since  $\gamma_{G_n}^n$  is the universal bundle ( $\ell \gg n$ ),  $c_{\sim}^{TN \oplus \varepsilon_N^3}$  is homotopic to  $c_{\alpha|N}^{TN} \oplus ((\alpha|N) \times id_{\mathbb{R}^3})$ , and  $t_{\Omega} \circ (c_{\alpha|N}^{TN} \oplus c_{\sim}^{\nu_N}) \circ t_N^{-1}$  is homotopic to  $(\alpha|N) \times id_{\mathbb{R}^{n+\ell}}$ . Hence,  $c_N$  is homotopic to  $\pi_{G_n}^k \circ \alpha|N$ . By [2, Proposition 3.3] again,  $c_{\sim}^{\nu_N}$  and  $c_{\alpha|N}^{\nu_N}$  are homotopic as bundle maps  $\nu_N \rightarrow \hat{\gamma}_{\Omega}^{\ell}$ . Since we have the bundle map

$$\mathbf{j}(c_{TN}, id_{TP})_{\Omega} : \Omega(TN, TP) \longrightarrow \Omega = \Omega(\gamma_{G_n}^n, TP)$$

covering  $c_N \times id_P$ , there is a section  $s_N : N \rightarrow \Omega(TN, TP)$  such that  $\pi_P^k \circ s_N = \pi_P^k \circ \alpha|N$  and  $\mathbf{j}(c_{TN}, id_{TP})_{\Omega} \circ s_N$  is homotopic to  $\alpha|N$ . Since  $\Omega(TN, TP)$  satisfies the h-principle, there exists an  $\Omega$ -regular map  $f : N \rightarrow P$  such that  $j^k f$  is homotopic

to  $s_N$ . This implies that  $\mathbf{j}(c_{TN}, id_{TP})\Omega \circ j^k f$  and  $\alpha|N$  are homotopic. This proves that

$$\begin{aligned}\omega([f]) &= [T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{\alpha|N}^{\nu_N}) \circ a_N] \\ &= [T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ T(c_{\sim}^{\nu_N}) \circ a_N] \\ &= [T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_*)}) \circ \alpha] \\ &= \tilde{\alpha}.\end{aligned}$$

This is what we want.  $\square$

Under the assumption of Theorem 1.1  $\mathfrak{C}(n, P; \Omega, \Omega_*)$  inherits the structure of an abelian group from the stable homotopy groups. The null element is defined to be represented by an  $\Omega$ -regular map  $f : N \rightarrow P$ , which has an  $\Omega_*$ -cobordism  $\mathfrak{C} : (W, \partial W) \rightarrow (P \times [0, 1], P \times 0)$  with  $\partial W = N$  such that  $\mathfrak{C}|N = f$  under the identification  $P \times 0 = P$ .

## 5. EXAMPLES OF $\Omega_*(n+1, p+1)$

In this section  $n < p$  is not necessarily assumed. As an important example of  $\Omega_*(n+1, p+1)$  for  $\Omega(n, p)$  we recall  $\Omega_{\mathcal{K}}^1 = \Omega_{\mathcal{K}}(n+1, p+1)$ , which is the set consisting of all  $\mathcal{K}$ -orbits  $\mathcal{K}(i_{+1}(z))$  for  $k$ -jets  $z \in \Omega(n, p)$ . Let  $\mathcal{C}_m$  denote the ring of smooth function germs  $(\mathbb{R}^m, 0) \rightarrow \mathbb{R}$  and let  $\mathfrak{m}_m$  denote its maximal ideal.

**Lemma 5.1.** *Let  $i$  be a non-negative integer smaller than  $n+1$ . Then any  $k$ -jet  $w \in \Sigma^i(n+1, p+1)$  has a  $k$ -jet  $z \in \Sigma^i(n, p)$  such that  $w$  lies in  $\mathcal{K}(i_{+1}(z))$ .*

*Proof.* We consider the usual coordinates  $x = (x_1, x_2, \dots, x_{n+1})$  of  $\mathbb{R}^{n+1}$  and  $y = (y_1, y_2, \dots, y_{p+1})$  of  $\mathbb{R}^{p+1}$ . Under suitable respective coordinates

$$x' = (x'_1, x'_2, \dots, x'_{n+1}) \quad \text{and} \quad y' = (y'_1, y'_2, \dots, y'_{p+1})$$

of  $\mathbb{R}^{n+1}$  and  $\mathbb{R}^{p+1}$ ,  $w$  is represented as  $w = j_0^k g$  with

$$(y'_1 \circ g(x'), \dots, y'_{p+1} \circ g(x')) = (x'_1, \dots, x'_{n-i}, g^{n-i+1}(x'), \dots, g^p(x'), x'_{n+1}),$$

where  $g^j \in \mathfrak{m}_{n+1}^2$ . Let  $\bar{g} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{p+1}$  be the map germ defined by

$$(y_1 \circ \bar{g}(x), \dots, y_{p+1} \circ \bar{g}(x)) = (x_1, \dots, x_{n-i}, g^{n-i+1}(x), \dots, g^p(x), x_{n+1}).$$

It is evident that  $j_0^k \bar{g} \in \mathcal{K}(w)$ . Let  $\overset{\circ}{x} = (x_1, \dots, x_n, 0)$ . Define the map germ  $f : \mathbb{R}^n = \mathbb{R}^n \times 0 \rightarrow \mathbb{R}^p$  by

$$f(\overset{\circ}{x}) = \begin{cases} (x_1, \dots, x_{n-i}, g^{n-i+1}(\overset{\circ}{x}), g^{n-i+2}(\overset{\circ}{x}), \dots, g^p(\overset{\circ}{x})) & \text{for } i < n, \\ (g^1(\overset{\circ}{x}), g^2(\overset{\circ}{x}), \dots, g^p(\overset{\circ}{x})) & \text{for } i = n. \end{cases}$$

Then we have

$$\begin{aligned}Q(j_0^k \bar{g}) &= \mathcal{C}_{n+1}/(\bar{g}^*(\mathfrak{m}_{p+1}) + \mathfrak{m}_{n+1}^{k+1}) \\ &\approx \mathcal{C}_n/(f^*(\mathfrak{m}_p) + \mathfrak{m}_n^{k+1}) \\ &= Q(j_0^k f).\end{aligned}$$

Setting  $z = j_0^k f$ , we have  $w \in \mathcal{K}(i_{+1}(z))$ .  $\square$

**Lemma 5.2.** *Let  $\Omega(n, p)$  be a  $\mathcal{K}$ -invariant open subset of  $J^k(n, p)$ . Then the set  $\Omega_{\mathcal{K}}^m$  is open in  $J^k(n+m, p+m)$  for  $m > 0$ .*

*Proof.* It is enough to prove the case  $m = 1$ . Suppose to the contrary that  $\Omega_{\mathcal{K}}^1$  is not open. Then there exists a  $k$ -jet  $w \in \Omega_{\mathcal{K}}^1$  and  $k$ -jets  $w_j \notin \Omega_{\mathcal{K}}^1$  such that  $\lim_{j \rightarrow \infty} w_j = w$ . By definition, there exists a  $k$ -jet  $z \in \Omega(n, p)$  such that  $w \in \mathcal{K}(i_{+1}(z))$ . Let  $i = n - \text{rank} z$ . Then we may assume without loss of generality that  $w = i_{+1}(z)$ ,  $w = j_0^k g$ ,  $w_j = j_0^k g_j$  and  $\lim_{j \rightarrow \infty} w_j = w$  with

$$\begin{aligned} (y_1 \circ g(x), \dots, y_{p+1} \circ g(x)) &= (x_1, \dots, x_{n-i}, g^{n-i+1}(x), \dots, g^p(x), x_{n+1}), \\ (y_1 \circ g_j(x), \dots, y_{p+1} \circ g_j(x)) &= (g_j^1(x), \dots, g_j^{n-i}(x), g_j^{n-i+1}(x), \dots, g_j^{p+1}(x)), \end{aligned}$$

where  $g^t \in \mathfrak{m}_n^2$  for  $n - i + 1 \leq t \leq p$ . Since  $\lim_{j \rightarrow \infty} w_j = i_{+1}(z)$ , we set

$$\begin{aligned} h_j^1(x_1, \dots, x_{n+1}) &= (x_1, \dots, x_{n-i}, g_j^{n-i+1}(x), \dots, g_j^p(x), x_{n+1}), \\ h_j^2(x_1, \dots, x_{n+1}) &= (x_1, \dots, x_{n-i}, g_j^{n-i+1}(\overset{\circ}{x}), \dots, g_j^p(\overset{\circ}{x}), x_{n+1}). \end{aligned}$$

Since the map germ defined by

$$(x_1, \dots, x_{n+1}) \longmapsto (g_j^1(x), \dots, g_j^{n-i}(x), x_{n-i+1}, \dots, x_n, g_j^{p+1}(x))$$

is a local diffeomorphism for sufficiently large numbers  $j$ , we have  $Q_k(w_j) \approx Q_k(j_0^k h_j^1) \approx Q_k(j_0^k h_j^2)$ . Let us define

$$f_j(x_1, \dots, x_n) = \begin{cases} (x_1, \dots, x_{n-i}, g_j^{n-i+1}(\overset{\circ}{x}), \dots, g_j^p(\overset{\circ}{x})) & \text{for } i < n, \\ (g_j^1(\overset{\circ}{x}), \dots, g_j^p(\overset{\circ}{x})) & \text{for } i = n. \end{cases}$$

Then we have that  $\lim_{j \rightarrow \infty} j_0^k h_j^1 = w = i_{+1}(z)$  and  $i_{+1}(j_0^k f_j) = j_0^k h_j^2$ , and hence,  $Q_k(w_j) \approx Q_k(j_0^k f_j)$ . Since  $\lim_{j \rightarrow \infty} j_0^k f_j = z$ , we have that  $j_0^k f_j \in \Omega(n, p)$  for sufficiently large numbers  $j$ . By definition, we have  $j_0^k h_j^2 \in \Omega_{\mathcal{K}}^1$  for sufficiently large numbers  $j$ . Since  $Q_k(w_j) \approx Q_k(j_0^k h_j^2)$ , it follows from [37, Theorem 2.1] that  $w_j \in \Omega_{\mathcal{K}}^1$  for sufficiently large numbers  $j$ . This is a contradiction.  $\square$

**Lemma 5.3.** *If two map germs  $f_1, f_2 : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^q, 0)$  are  $\mathcal{K}$ -equivalent, then the Boardman symbols of  $j_0^k f_1$  and  $j_0^k f_2$  are the same. Consequently, the Boardman manifold  $\Sigma^I(m, q)$  is invariant with respect to the action of  $\mathcal{K}$ .*

*Proof.* By [35] there exist a germ of a diffeomorphism  $h : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$  and a smooth map germ  $M : (\mathbb{R}^m, 0) \rightarrow GL(q)$  such that  $M(x)f_1(h(x)) = f_2(x)$ . Let  $f_i(x) = (f_1^i(x), \dots, f_q^i(x))$  with  $f_j^i \in \mathcal{C}_m/\mathfrak{m}_m^{k+1}$  ( $i = 1, 2$ ). Let  $\mathfrak{J}(f_i)$  denote the ideal of  $\mathcal{C}_m/\mathfrak{m}_m^{k+1}$  generated by  $f_1^i, \dots, f_q^i$ . Let  $h_* : \mathcal{C}_m/\mathfrak{m}_m^{k+1} \rightarrow \mathcal{C}_m/\mathfrak{m}_m^{k+1}$  be the isomorphism defined by  $h_*(\phi) = \phi \circ h$ . Then we have  $h_*(\mathfrak{J}(f_1)) = \mathfrak{J}(f_2)$ . The Boardman symbols of  $j_0^k f_i$  ( $i = 1, 2$ ) are determined by  $\mathfrak{J}(f_i)$  and are the same by [38]. This proves the assertion.  $\square$

**Lemma 5.4.** *If  $I$  is a Boardman symbol such that  $\Omega^I(n, p)$  is nonempty, then  $\Omega^I(n+1, p+1)$  is the union of all  $\mathcal{K}$ -orbits  $\mathcal{K}(i_{+1}(z))$  for  $z \in \Omega^I(n, p)$ .*

*Proof.* Let  $I = (i_1, i_2, \dots)$ . Since  $\Omega^I(n, p)$  is nonempty, we have  $i_1 \leq n$ . Let  $w \in \Omega^I(n+1, p+1)$ , whose Boardman symbol is  $J \leq I$ . Since  $j_1 \leq i_1 \leq n$ , it follows from Lemma 5.1 that there exists a  $z \in \Omega^I(n, p)$  such that  $w \in \mathcal{K}(i_{+1}(z))$ .

Conversely, let  $z \in \Omega^I(n, p)$  with Boardman symbol  $K \leq I$ . Since the Boardman symbol of  $i_{+1}(z)$  is obviously equal to  $K$ , we have  $i_{+1}(z) \in \Omega^I(n+1, p+1)$ . This shows the assertion.  $\square$

## 6. PRELIMINARIES FOR THEOREM 1.2

We prepare lemmas and propositions for the proof of Theorem 1.2.

Let  $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$  with 1 being the  $i$ -th component. Let  $pr_{p+1} : \mathbb{R}^{p+1} \rightarrow \mathbb{R}$  be the projection mapping  $(x_1, \dots, x_{p+1})$  to  $x_{p+1}$ . Let  $\pi_1^k : J^k(n, p) \rightarrow J^1(n, p)$  be the canonical forgetting projection.

Let  $K$  be a finite simplicial complex and  $L$  be its subcomplex such that  $K \setminus L$  is a manifold and  $\dim L < \dim K$ .

**Lemma 6.1.** *Let  $\Omega(n, p)$  be a  $\mathcal{K}$ -invariant open subset of  $J^k(n, p)$ . Let  $(K, L)$  be given as above and  $\dim K < p$ . Let  $\psi : (K, L) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  be a map such that  $\psi|(K \setminus L)$  is smooth. Then there exists a homotopy  $\psi_\lambda : (K, L) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  such that*

- (i)  $\psi_0 = \psi$ ,
- (ii)  $\psi_\lambda|L = \psi|L$ ,
- (iii)  $pr_{p+1}(\pi_1^k \circ \psi_1(u)(x_1, \dots, x_{n+1})) = x_{n+1}$  for any  $u \in K$ .

*Proof.* Let us define  $\mathbf{e} : K \rightarrow \mathbb{R}^{p+1}$  by  $\mathbf{e}(u) = (\pi_1^k \circ \psi(u))(\mathbf{e}_{n+1})$ . Since  $\psi(L) \subset i_{+1}(\Omega(n, p))$ , we have that, for any  $u \in L$ ,  $\mathbf{e}(u) = \mathbf{e}_{p+1}$ . Consider the fiber bundle  $\mathfrak{D}_{\mathbb{R}^{p+1}} : J^k(n+1, p+1) \rightarrow \mathbb{R}^{p+1}$  defined by  $\mathfrak{D}_{\mathbb{R}^{p+1}}(j_0^k f) = j_0^1 f(\mathbf{e}_{n+1})$ . Since  $\dim K < p$ ,  $K \setminus L$  is a manifold,  $\psi|(K \setminus L)$  is smooth and since  $\Omega_{\mathcal{K}}^1$  is an open subset, it follows from the transversality theorem and the covering homotopy property of  $\mathfrak{D}_{\mathbb{R}^{p+1}}$  that there exists a homotopy  $\varphi_\lambda : K \rightarrow J^k(n+1, p+1)$  relative to  $L$  with  $\varphi_0 = \psi$  such that

(6.1-i) the deformation  $\mathbf{u}_\lambda = \mathfrak{D}_{\mathbb{R}^{p+1}} \circ \varphi_\lambda$  of  $\mathbf{e}$  with  $\mathbf{u}_0 = \mathbf{e}$  satisfies that  $\mathbf{u}_1$  does not take the value of any non-positive multiple of  $\mathbf{e}_{p+1}$  and

(6.1-ii)  $\varphi_\lambda(K) \subset \Omega_{\mathcal{K}}^1$  for any  $\lambda$ .

In the following an element of  $GL(m)$  is regarded as a linear isomorphism of  $\mathbb{R}^m$  and  $E_m$  is the unit matrix of degree  $m$ . Let  $h_\lambda^1 : (K, L) \rightarrow (GL(p+1), E_{p+1})$  be the homotopy defined by  $h_\lambda^1(u) = ((1-\lambda) + \lambda/\|\mathbf{u}_1(u)\|)E_{p+1}$ . It follows from (6.1-i) that  $h_\lambda^1(u)(\mathbf{u}_1(u)) \in S^p$  and  $h_\lambda^1(u)(\mathbf{u}_1(u)) \neq -\mathbf{e}_{p+1}$  for any  $u \in K$ . By considering the rotation which is the identity on all points orthogonal to both  $\mathbf{u}_1(u)$  and  $\mathbf{e}_{p+1}$  and rotates the great circle through  $\mathbf{u}_1(u)$  and  $\mathbf{e}_{p+1}$  so as to carry  $\mathbf{u}_1(u)$  to  $\mathbf{e}_{p+1}$  along the shorter way (when  $\mathbf{u}_1(u) = \mathbf{e}_{p+1}$ , we consider  $E_{p+1}$ ), we have the homotopy  $h_\lambda^2 : (K, L) \rightarrow (SO(p+1), E_{p+1})$  relative to  $L$  such that  $h_0^2(u) = E_{p+1}$  and  $h_\lambda^2(u) \circ h_\lambda^1(u)(\mathbf{u}_1(u)) = \mathbf{e}_{p+1}$  for any  $u \in K$ . Let  $h_\lambda : (K, L) \rightarrow (GL(p+1), E_{p+1})$  be the homotopy defined by  $h_\lambda = h_{2\lambda}^1$  for  $0 \leq \lambda \leq 1/2$  and  $h_\lambda = h_{2\lambda-1}^2 \circ h_1^1$  for  $1/2 \leq \lambda \leq 1$ . Define  $\kappa_{p+1} : K \rightarrow J^1(n+1, 1)$  by

$$\kappa_{p+1}(u) = pr_{p+1} \circ \pi_1^k(j^k(h_1(u)) \circ \varphi_1(u)).$$

Since  $\kappa_{p+1}(u)$  is of rank 1 for any  $u \in K$ , we have the unique vector  $V(u) \in \mathbb{R}^{n+1}$  of length 1 such that  $V(u)$  is perpendicular to  $\text{Ker}(\kappa_{p+1}(u))$  and  $\kappa_{p+1}(u)(V(u))$  is positive. Namely,  $h_1(u) \circ \pi_1^k(\varphi_1(u))(V(u))$  is directed to the same orientation of  $\mathbf{e}_{p+1}$ . Since  $\kappa_{p+1}(u)(\mathbf{e}_{n+1}) = 1$ ,  $V(u)$  cannot be equal to  $-\mathbf{e}_{n+1}$ .

We set  $v(u) = \kappa_{p+1}(u)(V(u)) > 0$ . By considering the rotation which is the identity on all points orthogonal to both  $V(u)$  and  $\mathbf{e}_{n+1}$  and rotates the great circle through  $V(u)$  and  $\mathbf{e}_{n+1}$  so as to carry  $\mathbf{e}_{n+1}$  to  $V(u)$  along the shorter way, we again have the homotopy  $H_\lambda^1 : (K, L) \rightarrow (SO(n+1), E_{n+1})$  relative to  $L$  such that  $H_0^1(u) = E_{n+1}$  and  $H_\lambda^1(u)(\mathbf{e}_{n+1}) = V(u)$  for any  $u \in K$ . Let  $H_\lambda^2 : (K, L) \rightarrow (GL(n+1), E_{n+1})$  be the homotopy relative to  $L$  defined by  $H_\lambda^2(u) = ((1-\lambda) +$

$\lambda/v(u))E_{n+1}$ . Let  $H_\lambda : (K, L) \rightarrow (GL(n+1), E_{n+1})$  be the homotopy defined by  $H_\lambda(u) = H_{2\lambda}^1(u)$  for  $0 \leq \lambda \leq 1/2$  and  $H_\lambda(u) = H_{2\lambda-1}^2(u) \circ H_1^1(u)$  for  $1/2 \leq \lambda \leq 1$ . Then we have that for any  $u \in K$ ,

$$\begin{aligned} \kappa_{p+1}(u) \circ H_1(u)(\mathbf{e}_{n+1}) &= \kappa_{p+1}(u) \circ H_1^2(u) \circ H_1^1(u)(\mathbf{e}_{n+1}) \\ &= \kappa_{p+1}(u) \circ H_1^2(u)(V(u)) \\ &= \kappa_{p+1}(u)(V(u))/v(u) \\ &= 1. \end{aligned}$$

Since  $H_1^1(u) \in SO(n+1)$  and  $\mathbf{e}_i$  is orthogonal to  $\mathbf{e}_{n+1}$  ( $i < n+1$ ),  $H_1^1(u)(\mathbf{e}_i)$  is orthogonal to  $H_1^1(u)(\mathbf{e}_{n+1}) = V(u)$ . Namely,  $H_1(u)(\mathbf{e}_i)$  lies in  $\text{Ker}(\kappa_{p+1}(u))$ . Hence, we have

$$\kappa_{p+1}(u) \circ H_1(u)(\mathbf{e}_i) = 0 \quad \text{for } i < n+1.$$

Define the homotopy  $\psi_\lambda : (K, L) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  relative to  $L$  by

$$\psi_\lambda(u) = \begin{cases} \varphi_{3\lambda}(u) & \text{for } 0 \leq \lambda \leq 1/3, \\ h_{3\lambda-1}(u) \circ \varphi_1(u) & \text{for } 1/3 \leq \lambda \leq 2/3, \\ h_1(u) \circ \varphi_1(u) \circ H_{3\lambda-2}(u) & \text{for } 2/3 \leq \lambda \leq 1. \end{cases}$$

By the definition we have

$$pr_{p+1} \circ \pi_1^k(\psi_1(u))(\mathbf{e}_i) = \begin{cases} 0 & \text{for } i < n+1, \\ 1 & \text{for } i = n+1. \end{cases}$$

This is what we want.  $\square$

**Proposition 6.2.** *Under the same assumption of Lemma 6.1, we have a homotopy  $\Psi_\lambda : (K, L) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  such that*

- (i)  $\Psi_0 = \psi$ ,
- (ii)  $\Psi_\lambda|_L = \psi|_L$ ,
- (iii)  $\Psi_1(K) \subset i_{+1}(\Omega(n, p))$ .

*Proof.* Let  $\psi_\lambda$  be the homotopy given in Lemma 6.1. Let us express  $\psi_\lambda(u) = (f_\lambda^1(u), f_\lambda^2(u), \dots, f_\lambda^{p+1}(u))$  using the coordinates of  $\mathbb{R}^{p+1}$ , where  $f_\lambda^i(u)$  is regarded as a polynomial of degree at most  $k$  with constant 0. We note that

$$f_1^{p+1}(u)(x_1, \dots, x_{n+1}) = x_{n+1} + \text{higher term}.$$

Let  $\text{Diff}(\mathbb{R}^{n+1}, 0)$  be the space of all germs of local diffeomorphisms of  $(\mathbb{R}^{n+1}, 0)$ . Let us define a homotopy of maps  $\Phi_\lambda : (K, L) \rightarrow (\text{Diff}(\mathbb{R}^{n+1}, 0), id_{(\mathbb{R}^{n+1}, 0)})$  by

$$\Phi_\lambda(u)(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1} + \lambda(f_1^{p+1}(u) - x_{n+1})).$$

It is obvious that  $\Phi_\lambda(u)$  is a germ of a diffeomorphism of  $(\mathbb{R}^{n+1}, 0)$ . Then we have the inverse  $\Phi_\lambda(u)^{-1}$  such that

$$(6.1) \quad pr_{p+1} \circ \psi_1(u) \circ \Phi_1(u)^{-1}(x_1, \dots, x_{n+1}) = x_{n+1}.$$

We now define  $\phi_\lambda : (K, L) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  by

$$\phi_\lambda(u) = \psi_1 \circ j_0^k(\Phi_\lambda(u)^{-1}).$$

In order to exclude the terms containing  $x_{n+1}$  from  $y_j \circ \phi_1$  ( $1 \leq j \leq p$ ) we define the homotopy  $\eta_\lambda : (K, L) \rightarrow (J^k(n+1, p+1), i_{+1}(J^k(n, p)))$  by

$$(6.2) \quad \begin{aligned} \eta_\lambda(u)(x) &= (1-\lambda)\phi_1(u)(x_1, \dots, x_{n+1}) \\ &\quad + \lambda(\phi_1(u)(x_1, \dots, x_n, 0) + (0, \dots, 0, x_{n+1})). \end{aligned}$$

It is obvious that  $\eta_1(K) \subset i_{+1}(\Omega(n, p))$  and that  $\eta_\lambda|L = \psi|L$ . It remains to prove that  $\eta_\lambda$  is a homotopy to  $\Omega_{\mathcal{K}}^1$ . It follows from (6.1) and (6.2) that

$$pr_{p+1} \circ \eta_\lambda(u)(x) = (1 - \lambda)(x_{n+1}) + \lambda x_{n+1} = x_{n+1}.$$

Let us express  $\eta_\lambda(u) = (g_\lambda^1(u), g_\lambda^2(u), \dots, g_\lambda^{p+1}(u))$ , where  $g_\lambda^i(u)$  is regarded as a polynomial of degree at most  $k$  with constant 0. Consider the ideal  $\mathfrak{I}_\lambda(u)$  generated by  $g_\lambda^1(u), g_\lambda^2(u), \dots, g_\lambda^{p+1}(u)$  in  $\mathfrak{m}_{n+1}/\mathfrak{m}_{n+1}^{k+1}$ . Then  $\mathfrak{I}_\lambda(u)$  is constantly equal to  $\mathfrak{I}_0(u)$ , and hence  $Q(\eta_\lambda(u)) \approx Q(\psi_1(u))$ . Since  $\psi_1(u) \in \Omega_{\mathcal{K}}^1$ , we have  $\eta_\lambda(u) \in \Omega_{\mathcal{K}}^1$  by [37]. Then the required homotopy  $\Psi_\lambda$  is defined by  $\Psi_\lambda = \psi_{3\lambda}$  ( $0 \leq \lambda \leq 1/3$ ),  $\Psi_\lambda = \phi_{3\lambda-1}$  ( $1/3 \leq \lambda \leq 2/3$ ) and  $\Psi_\lambda = \eta_{3\lambda-2}$  ( $2/3 \leq \lambda \leq 1$ ).  $\square$

**Proposition 6.3.** *Let  $\Omega(n, p)$  be a  $\mathcal{K}$ -invariant open subset of  $J^k(n, p)$ . Then  $i_{+1} : \Omega(n, p) \rightarrow \Omega_{\mathcal{K}}^1$  is a homotopy  $(p-1)$ -equivalence.*

*Proof.* Let  $\iota_n$  denote a jet of  $\Omega(n, p)$  and  $\iota_{n+1} = i_{+1}(\iota_n)$ . We first prove that  $(i_{+1})_* : \pi_i(\Omega(n, p)) \rightarrow \pi_i(\Omega_{\mathcal{K}}^1)$  is surjective for  $0 \leq i \leq p-1$ . Indeed, let  $[a] \in \pi_i(\Omega_{\mathcal{K}}^1)$  be represented by  $a : (S^i, \mathbf{e}_1) \rightarrow (\Omega_{\mathcal{K}}^1, \iota_{n+1})$ . Then by Proposition 6.2 we have a homotopy  $\varphi_\lambda : (S^i, \mathbf{e}_1) \rightarrow (\Omega_{\mathcal{K}}^1, \iota_{n+1})$  such that  $\varphi_1(S^i) \subset i_{+1}(\Omega(n, p))$ .

Next let  $0 \leq i < p-1$ . Let  $[b] \in \pi_i(\Omega(n, p))$  be represented by  $b : (S^i, \mathbf{e}_1) \rightarrow (\Omega(n, p), \iota_n)$  such that  $(i_{+1})_*([b]) = 0$ . Then we have a homotopy  $\tilde{\varphi} : S^i \times [0, 1] \rightarrow (\Omega_{\mathcal{K}}^1, \iota_{n+1})$  such that  $\tilde{\varphi}|S^i \times 0 = i_{+1} \circ b$  under the identification  $S^i = S^i \times 0$  and  $\tilde{\varphi}(\mathbf{e}_1 \times [0, 1] \cup S^i \times 1) = \iota_{n+1}$ . It follows from Proposition 6.2 that since  $i+1 < p$ , there exists a homotopy  $\Phi_\lambda : (S^i \times [0, 1], S^i \times 0 \cup \mathbf{e}_1 \times [0, 1] \cup S^i \times 1) \rightarrow (\Omega_{\mathcal{K}}^1, i_{+1}(\Omega(n, p)))$  relative to  $S^i \times 0 \cup \mathbf{e}_1 \times [0, 1] \cup S^i \times 1$  such that  $\Phi_1(S^i \times [0, 1]) \subset i_{+1}(\Omega(n, p))$ . This proves the injectivity of  $(i_{+1})_* : \pi_i(\Omega(n, p)) \rightarrow \pi_i(\Omega_{\mathcal{K}}^1)$ .  $\square$

## 7. PROOF OF THEOREM 1.2

Let  $A_{m,q}$  express  $G_{m,q}$  or  $\tilde{G}_{m,q}$ . Let  $i_{A_{m+r}} : A_{m,q} \rightarrow A_{m+r,q}$  denote the injection mapping an  $m$ -plane  $a$  to the  $(m+r)$ -plane including  $a$  and the canonical vectors  $\mathbf{e}_{q+m+1}, \dots, \mathbf{e}_{q+m+r}$  in  $\mathbb{R}^{q+m+r}$ . We use the notation  $\Omega_{\mathcal{K}}^m = \Omega_{\mathcal{K}}(\gamma_{G_{n+m}}^{n+m}, TP \oplus \varepsilon_P^m)$ .

**Lemma 7.1.** *The map  $i_{A_{m+r}} : A_{m,q} \rightarrow A_{m+r,q}$  is a homotopy  $m$ -equivalence.*

*Proof.* We only prove the unoriented case. The proof in the oriented case is similar. Let us consider the diagram with the canonical maps as described

$$\begin{array}{ccc} A_{m,q} & \xrightarrow{\rho_0} & O(q+m+r)/O(q) \times O(m) \times E_r \\ i_{A_{m+r}} \downarrow & \rho_2 \swarrow & \downarrow \rho_1 \\ A_{m+r,q} & & O(q+m+r)/O(q+m) \times E_r, \end{array}$$

where  $\rho_0$  is induced from the inclusion  $\mathbb{R}^{q+m} = \mathbb{R}^{q+m} \times 0 \rightarrow \mathbb{R}^{q+m+r}$  and  $\rho_1$  and  $\rho_2$  are induced from the inclusions  $O(q) \times O(m) \rightarrow O(q+m)$  and  $O(m) \times E_r \rightarrow O(m+r)$  respectively. Since  $A_{m,q}$  is a fiber of the fiber bundle  $\rho_1$ ,  $\rho_0$  is a homotopy  $(q+m-1)$ -equivalence. Since  $\rho_2$  is a homotopy  $m$ -equivalence,  $i_{A_{m+r}}$  is also a homotopy  $m$ -equivalence.  $\square$

In the following lemma  $p$  is not necessarily larger than  $n$ .

**Lemma 7.2.** *Let  $\Omega(n, p)$  be a  $\mathcal{K}$ -invariant open subset of  $J^k(n, p)$ . Then the fiber map  $\Delta^{(\Omega, \Omega_{\mathcal{K}})} : \Omega \rightarrow \Omega_{\mathcal{K}}^1$  is a homotopy  $\min\{n, p-1\}$ -equivalence.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & \pi_i(\Omega(n, p)) & \xrightarrow{\partial} & \pi_i(\Omega) & \longrightarrow & \pi_i(G_{n, \ell} \times P) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \pi_i(\Omega_{\mathcal{K}}^1) & \xrightarrow{\partial} & \pi_i(\Omega_{\mathcal{K}}^1) & \longrightarrow & \pi_i(G_{n+1, \ell} \times P) & \longrightarrow \end{array}$$

which is induced from the homomorphisms  $(\Delta^{(\Omega, \Omega_{\mathcal{K}})})_*$  of the exact sequence of the homotopy groups for the fiber bundle  $\Omega$  over  $G_{n, \ell} \times P$  to that for the fiber bundle  $\Omega_{\mathcal{K}}^1$  over  $G_{n+1, \ell} \times P$ . Then it follows from Lemma 7.1 for  $(i^G)_*$  and Proposition 6.3 that if  $0 \leq i < \min\{n, p-1\}$ , then  $(\Delta^{(\Omega, \Omega_{\mathcal{K}})})_* : \pi_i(\Omega) \rightarrow \pi_i(\Omega_{\mathcal{K}}^1)$  is an isomorphism by the five lemma and if  $i = \min\{n, p-1\}$ , then it is an epimorphism by [13, Lemma 3.2].  $\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* We only prove the unoriented case. In the oriented case we only need the Thom Isomorphism Theorem under the coefficient group  $\mathbb{Z}$ . It follows from Lemma 7.2 and the Whitehead theorem ([48, Section 5, 9 Theorem]) that

$$(\Delta^{(\Omega, \Omega_{\mathcal{K}})})_* : H_i(\Omega) \rightarrow H_i(\Omega_{\mathcal{K}}^1)$$

is an isomorphism for  $0 \leq i < n$  and an epimorphism for  $i = n$ .

Let  $\ell \gg n, p$ . By virtue of the Thom Isomorphism Theorem, we have that

$$T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_{\mathcal{K}})})_* : H_{i+\ell}(T(\hat{\gamma}_{\Omega}^{\ell}); \mathbb{Z}/(2)) \longrightarrow H_{i+\ell}(T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}); \mathbb{Z}/(2))$$

is an isomorphism for  $-\ell \leq i < n$  and an epimorphism for  $i = n$ .

Let  $\mathcal{C}$  denote the Serre class of finite groups of orders prime to two. Then it follows from [45, Proposition 2, p. 277] that

$$(7.1) \quad T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_{\mathcal{K}})})_* : H_{i+\ell}(T(\hat{\gamma}_{\Omega}^{\ell}); \mathbb{Z}) \longrightarrow H_{i+\ell}(T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}); \mathbb{Z})$$

is a  $\mathcal{C}$ -isomorphism for  $-\ell \leq i < n$  and a  $\mathcal{C}$ -epimorphism for  $i = n$ . By the Whitehead theorem modulo  $\mathcal{C}$  ([45, Theorem 3, p. 276]),

$$T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_{\mathcal{K}})})_* : \pi_{i+\ell}(T(\hat{\gamma}_{\Omega}^{\ell})) \longrightarrow \pi_{i+\ell}(T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}))$$

is a  $\mathcal{C}$ -isomorphism for  $-\ell \leq i < n$  and a  $\mathcal{C}$ -epimorphism for  $i = n$ .

Let  $\mathfrak{N}_{\Omega}^i$  (respectively  $\mathfrak{N}_{\Omega_{\mathcal{K}}^1}^i$ ) denote the cobordism group of smooth maps  $s$  of closed  $i$ -manifolds  $M$  to  $\Omega$  (respectively  $\Omega_{\mathcal{K}}^1$ ) under the corresponding cobordism of smooth maps such that there exists a bundle map of the stable  $\ell$ -dimensional normal bundle  $\nu_M$  to  $\hat{\gamma}_{\Omega}^{\ell}$  (respectively  $\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}$ ) covering  $s$ . It follows from the standard argument in the cobordism theory (see, for example, [50]) that

$$\mathfrak{N}_{\Omega}^i \approx \pi_{i+\ell}(T(\hat{\gamma}_{\Omega}^{\ell})) \quad \text{and} \quad \mathfrak{N}_{\Omega_{\mathcal{K}}^1}^i \approx \pi_{i+\ell}(T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell})).$$

Consequently, any element of  $\pi_{i+\ell}(T(\hat{\gamma}_{\Omega}^{\ell}))$  and  $\pi_{i+\ell}(T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}))$  is of order two. This together with (7.1) proves Theorem 1.2.  $\square$

By Lemmas 5.2 and 5.4 it will be easy to see inductively that  $\Omega_{\mathcal{K}}(n+m, p+m)$  is the subset of  $J^k(n+m, p+m)$  which consists of all  $\mathcal{K}$ -orbits  $\mathcal{K}(i_{+1}(z))$  for  $k$ -jets  $z \in \Omega_{\mathcal{K}}(n+m-1, p+m-1)$  for  $m > 1$  and that if  $\Omega(n, p) = \Omega^I(n, p) \neq \emptyset$ , then we have  $\Omega^I(n+m, p+m) = \Omega_{\mathcal{K}}(n+m, p+m)$ . By using the map  $J^k(n+m, p+m) \rightarrow J^k(n+m+q, p+m+q)$  sending a jet  $j_0^k f$  to  $j_0^k(f \times id_{\mathbb{R}^q})$ , we have the canonical

map  $\Omega_{\mathcal{K}}^m \rightarrow \Omega_{\mathcal{K}}^{m+q}$ . This induces the canonical bundle map  $\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^m, \Omega_{\mathcal{K}}^{m+q})} : \hat{\gamma}_{\Omega_{\mathcal{K}}^m}^\ell \rightarrow \hat{\gamma}_{\Omega_{\mathcal{K}}^{m+q}}^\ell$  and the associated map  $T(\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^m, \Omega_{\mathcal{K}}^{m+q})}) : T(\hat{\gamma}_{\Omega_{\mathcal{K}}^m}^\ell) \rightarrow T(\hat{\gamma}_{\Omega_{\mathcal{K}}^{m+q}}^\ell)$ . Let  $\theta$  denote an integer such that  $\theta = 1$  when  $n < p$  and  $\theta = n + 2 - p$  when  $n \geq p \geq 2$ .

**Proposition 7.3.** *Let  $\theta$  be the integer as above. Then the homomorphism*

$$\lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell) \right) \longrightarrow \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^{\theta+q}}^\ell) \right)$$

*induced from the above map  $T(\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^\theta, \Omega_{\mathcal{K}}^{\theta+q})})$  is an isomorphism for  $q \geq 0$ .*

*Proof.* We only prove the unoriented case. The proof proceeds as in the Proof of Theorem 1.2. By the iterated use of Lemma 7.2, we have that  $T(\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^\theta, \Omega_{\mathcal{K}}^{\theta+q})})$  is a homotopy  $\min\{n + \theta, p + \theta - 1\}$ -equivalence, and hence, a homotopy  $(n + 1)$ -equivalence. Therefore,

$$(T(\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^\theta, \Omega_{\mathcal{K}}^{\theta+q})}))_* : H_i(\Omega_{\mathcal{K}}^\theta) \longrightarrow H_i(\Omega_{\mathcal{K}}^{\theta+q})$$

is an isomorphism for  $0 \leq i \leq n$  and an epimorphism for  $i = n + 1$ . By the argument similar to that in the Proof of Theorem 1.2, we have that

$$(7.2) \quad (T(\mathbf{b}(\hat{\gamma})^{(\Omega_{\mathcal{K}}^\theta, \Omega_{\mathcal{K}}^{\theta+q})}))_* : \pi_{i+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell) \right) \longrightarrow \pi_{i+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^{\theta+q}}^\ell) \right)$$

is a  $\mathcal{C}$ -isomorphism for  $-\ell \leq i \leq n$  and a  $\mathcal{C}$ -epimorphism for  $i = n + 1$ . Since any element of the groups in (7.2) is of order two, we obtain the proposition.  $\square$

## 8. CLASSIFYING SPACE

In this section we will induce the classifying space  $B_{\mathfrak{D}}$  in Theorem 1.3.

We consider the vector bundle  $J^k(\gamma_{G_{n+m}}^{n+m}, \mathbb{R}^{p+m})$  with the projection  $\pi_{G_{n+m}}^k$  onto  $G_{n+m} := G_{n+m} \times \{\text{a point}\}$  and the open subbundle  $\Omega_{\mathcal{K}}(\gamma_{G_{n+m}}^{n+m}, \mathbb{R}^{p+m})$  of  $J^k(\gamma_{G_{n+m}}^{n+m}, \mathbb{R}^{p+m})$  associated to  $\Omega_{\mathcal{K}}^m$ , where  $\mathbb{R}^{p+m}$  is regarded as a vector bundle over a point. As in Section 3, the spaces  $\{T((\pi_{G_m}^k)^*(\hat{\gamma}_{G_m}^\ell)|_{\Omega_{\mathcal{K}}})\}_\ell$  constitute a spectrum.

Let  $\varkappa$  denote any integer with  $\varkappa \geq p + 3$ . Let  $\ell \gg n, p, \varkappa$ . We set  $\mathbb{G}_{n+\theta+\varkappa} = A_{n+\theta+\varkappa, \ell+p}$ . Let  $P$  be embedded in  $\mathbb{R}^{p+\varkappa}$ . Let  $x \in G_{n+\theta}$ ,  $y \in P$  and  $(\nu_P)_y$  be the orthogonal complement of  $T_y P$  in  $\mathbb{R}^{p+\varkappa}$ . Let  $c : G_{n+\theta} \times P \rightarrow \mathbb{G}_{n+\theta+\varkappa}$  denote the map such that  $c(x, y)$  is the  $(n + \theta + \varkappa)$ -subspace  $x \oplus (\nu_P)_y$  in  $\mathbb{R}^{n+\theta+\varkappa+\ell+p}$ . Let  $\mathbf{c} : \pi_{G_{n+\theta}}^*(\gamma_{G_{n+\theta}}^{n+\theta}) \oplus \pi_P^*(\nu_P) \rightarrow \gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}$  be the bundle map which is canonically induced to cover the classifying map  $c$ .

Let  $\mathbf{J}^k(\gamma^{n+\theta} \oplus \nu, TP \oplus \varepsilon_P^\theta \oplus \nu_P)$  denote

$$\text{Hom} \left( \bigoplus_{i=1}^k S^i(\pi_{G_{n+\theta}}^*(\gamma_{G_{n+\theta}}^{n+\theta}) \oplus \pi_P^*(\nu_P)), \pi_P^*(TP \oplus \varepsilon_P^\theta \oplus \nu_P) \right)$$

and define the fiber map

$$(8.1) \quad J^k(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta) \longrightarrow \mathbf{J}^k(\gamma^{n+\theta} \oplus \nu, TP \oplus \varepsilon_P^\theta \oplus \nu_P)$$

over  $G_{n+\theta} \times P$  by mapping  $j_x^k \alpha \in J_{x,y}^k(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta)$  to  $j_{(x,y)}^k(\alpha \times id_{(\nu_P)_y})$ . We also obtain the bundle map

$$(8.2) \quad \mathbf{J}^k(\gamma^{n+\theta} \oplus \nu, TP \oplus \varepsilon_P^\theta \oplus \nu_P) \longrightarrow J^k(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, TP \oplus \varepsilon_P^\theta \oplus \nu_P)$$



covering  $(c, \pi_P) : G_{n+\theta} \times P \rightarrow \mathbb{G}_{n+\theta+\varkappa} \times P$  which is canonically induced from the bundle map  $\mathbf{c}$  and  $id_{TP \oplus \varepsilon_P^\theta \oplus \nu_P}$ . It follows from Lemma 7.1 that  $(c, \pi_P)$  is a homotopy  $(n+\theta)$ -equivalence. The composite of the maps in (8.1) and (8.2) on the fibers over  $(x, y)$  and  $(c(x, y), y)$  induces a map

$$\Omega_{\mathcal{K}}((\gamma_{G_{n+\theta}}^{n+\theta})_x, T_y P \oplus \varepsilon_y^\theta) \longrightarrow \Omega_{\mathcal{K}}((\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa})_{c(x,y)}, T_y P \oplus \varepsilon_y^\theta \oplus (\nu_P)_y),$$

which is a homotopy  $(n+1)$ -equivalence by the iterated use of Proposition 6.3 and  $G_{n+\theta} \times P \rightarrow \mathbb{G}_{n+\theta+\varkappa} \times P$  is a homotopy  $(n+\theta)$ -equivalence by Lemma 7.1. Consequently, we obtain the fiber map

$$(8.3) \quad \mathbf{j}_{\Omega_{\mathcal{K}}} : \Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta) \longrightarrow \Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, TP \oplus \varepsilon_P^\theta \oplus \nu_P)$$

covering  $(c, \pi_P)$ , which is also a homotopy  $(n+1)$ -equivalence.

Trivialization  $TP \oplus \nu_P \rightarrow \varepsilon_P^{p+\varkappa}$  induces the bundle isomorphism

$$J^k(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, TP \oplus \varepsilon_P^\theta \oplus \nu_P) \longrightarrow J^k(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa}) \times P$$

over  $\mathbb{G}_{n+\theta+\varkappa} \times P$ , where  $\mathbb{R}^{p+\theta+\varkappa}$  is regarded as the trivial vector bundle over a point. Let  $\Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})$  be the open subbundle associated to  $\Omega_{\mathcal{K}}^{\theta+\varkappa}$ . Then we have the bundle map

$$(8.4) \quad \mathbf{k}_{\Omega_{\mathcal{K}}} : \Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, TP \oplus \varepsilon_P^\theta \oplus \nu_P) \longrightarrow \Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa}) \times P$$

over  $\mathbb{G}_{n+\theta+\varkappa} \times P$ . In the following we identify the two spaces in (8.4). Thus we have the following lemma.

**Lemma 8.1.** *The fiber map  $\mathbf{k}_{\Omega_{\mathcal{K}}} \circ \mathbf{j}_{\Omega_{\mathcal{K}}}$  covering  $(c, \pi_P)$  is a homotopy  $(n+1)$ -equivalence.*

Let  $\pi_{\mathbb{G}_{n+\theta+\varkappa}}^k : J^k(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa}) \rightarrow \mathbb{G}_{n+\theta+\varkappa}$  be the canonical projection. Let  $B(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P}$  denote the vector bundle over  $\Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa}) \times P$  defined by

$$B(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P} = (\pi_{\mathbb{G}_{n+\theta+\varkappa}}^k)^*(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p})|_{\Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})} \times \nu_P.$$

This satisfies

$$T(B(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P}) = T((\pi_{\mathbb{G}_{n+\theta+\varkappa}}^k)^*(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p})|_{\Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge T(\nu_P).$$

Let

$$\begin{aligned} \mathbf{b}_{\mathbf{k}_{\Omega_{\mathcal{K}}} \circ \mathbf{j}_{\Omega_{\mathcal{K}}}} &: (\pi_{G_{n+\theta}}^k)^*(\widehat{\gamma}_{G_{n+\theta}}^\ell) \oplus (\pi_P^k)^*(TP)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, \mathbb{R}^{p+\theta})} \\ &\longrightarrow ((\pi_{\mathbb{G}_{n+\theta+\varkappa}}^k)^*(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p})|_{\Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \times P \end{aligned}$$

be the bundle map covering  $\mathbf{k}_{\Omega_{\mathcal{K}}} \circ \mathbf{j}_{\Omega_{\mathcal{K}}}$  mapping  $(\gamma_{G_{n+\theta}}^{n+\theta})_x^\perp \oplus T_y P$  to  $((\gamma_{\mathbb{G}_{n+\theta}}^{n+\theta})_x \oplus (\nu_P)_y)^\perp$ , where  $\perp$  denotes the orthogonal complement. This induces a bundle map

$$\begin{aligned} \mathbf{B} &: ((\pi_{G_{n+\theta}}^k)^*(\widehat{\gamma}_{G_{n+\theta}}^\ell) \oplus (\pi_P^k)^*(TP \oplus \nu_P))|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta)} \\ &\longrightarrow B(\widehat{\gamma}_{\mathbb{G}_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P} \end{aligned}$$

covering

$$\mathbf{k}_{\Omega_{\mathcal{K}}} \circ \mathbf{j}_{\Omega_{\mathcal{K}}} : \Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta) \rightarrow \Omega_{\mathcal{K}}(\gamma_{\mathbb{G}_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa}) \times P,$$

which is a bundle map over  $P$  and is a homotopy  $(n+1)$ -equivalence.

Let  $X$  and  $Y$  be connected polyhedra with base points respectively. Let  $\{X; Y\}$  denote the set of  $S$ -homotopy classes of  $S$ -maps. Let  $G_{n+\theta+\varkappa} = A_{n+\theta+\varkappa, \ell}$  as in Introduction.

**Proposition 8.2.** *Let  $\Omega(n, p)$  denote a nonempty  $\mathcal{K}$ -invariant open subset. Then there exists an isomorphism*

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta)})^\ell \right) &\longrightarrow \\ \lim_{\ell \rightarrow \infty} \pi_{n+\ell+\varkappa} \left( T((\pi_{G_{n+\theta+\varkappa}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge T(\nu_P)) \right). \end{aligned}$$

*Proof.* Setting  $\Omega_{\mathcal{K}}^\theta = \Omega_{\mathcal{K}}(\gamma_{G_{n+\theta}}^{n+\theta}, TP \oplus \varepsilon_P^\theta)$ , we have that

$$\begin{aligned} (8.5) \quad \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell) \right) &\approx \{S^{n+\ell}; T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell)\} \\ &\approx \{S^{n+\ell+p+\varkappa}; T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell) \wedge S^{p+\varkappa}\} \\ &\approx \{S^{n+\ell+p+\varkappa}; T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell \oplus \varepsilon_{\Omega_{\mathcal{K}}^\theta}^{p+\varkappa})\} \\ &\approx \{S^{n+\ell+p+\varkappa}; T(\hat{\gamma}_{\Omega_{\mathcal{K}}^\theta}^\ell \oplus (\pi_P^k)^*(TP \oplus \nu_P)|_{\Omega_{\mathcal{K}}^\theta})\}. \end{aligned}$$

By Lemma 8.1 and an argument as in the proof of Theorem 1.2 using the Thom isomorphism Theorem, the associated map  $T(\mathbf{B})$  induces the isomorphism between the last group in (8.5) and  $\{S^{n+\ell+p+\varkappa}; T(B(\hat{\gamma}_{G_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P})\}$ . Furthermore, we have that

$$\begin{aligned} (8.6) \quad &\{S^{n+\ell+p+\varkappa}; T(B(\hat{\gamma}_{G_{n+\theta+\varkappa}}^{\ell+p}, \nu_P)_{\Omega_{\mathcal{K}} \times P})\} \\ &\approx \{S^{n+\ell+p+\varkappa}; T((\pi_{G_{n+\theta+\varkappa}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa}}^{\ell+p})|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge T(\nu_P))\} \\ &\approx \{S^{n+\ell+\varkappa}; T((\pi_{G_{n+\theta+\varkappa}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge T(\nu_P))\}. \end{aligned}$$

This proves the proposition.  $\square$

In the rest of this section we work in the oriented case and  $P$  should be connected and oriented. Let  $P^0$  be the union of  $P$  and the base point. Consider the duality map  $T(\nu_P) \wedge S^t(P^0) \rightarrow S^{p+\varkappa+t}$  in [47]. If  $P$  is connected in addition, then the last group in (8.6) in the oriented case is isomorphic to

$$(8.7) \quad \{S^{n+\ell+\varkappa} \wedge S^t(P^0); T((\pi_{G_{n+\theta+\varkappa, \ell}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa, \ell}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa, \ell}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge S^{p+\varkappa+t})\}.$$

Take a representative map  $\alpha$  in a homotopy class in (8.7) and consider the correspondence of a point  $y$  in  $P$  to a map  $\alpha|_{(S^{n+\ell+\varkappa+t} \wedge \{y, \text{base point}\})}$ . It is not difficult to see that the set of  $S$ -homotopy classes in (8.7) is bijective to the following sets of homotopy classes

$$\begin{aligned} &[P, C^0(S^{n+\ell+\varkappa+t}, T((\pi_{G_{n+\theta+\varkappa, \ell}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa, \ell}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa, \ell}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge S^{p+\varkappa+t})] \\ &\approx [P, C^0(S^{\ell+n-p}, T((\pi_{G_{n+\theta+\varkappa, \ell}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa, \ell}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa, \ell}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}))]. \end{aligned}$$

Setting

$$B_{\mathfrak{D}}^x = \lim_{\ell \rightarrow \infty} C^0(S^{\ell+n-p}, T((\pi_{G_{n+\theta+\varkappa, \ell}}^k)^*(\hat{\gamma}_{G_{n+\theta+\varkappa, \ell}}^\ell)|_{\Omega_{\mathcal{K}}(\gamma_{G_{n+\theta+\varkappa, \ell}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})})),$$

we define the classifying space  $B_{\mathcal{D}} = \lim_{\varkappa \rightarrow \infty} B_{\mathcal{D}}^{\varkappa}$  as in Introduction. We have the following proposition.

**Proposition 8.3.** *Let  $\Omega(n, p)$  be as in Proposition 8.2. Let  $P$  be a closed connected and oriented  $p$ -dimensional manifold. Then there exists an isomorphism*

$$\lim_{\ell \rightarrow \infty} \pi_{n+\ell+\varkappa} \left( T((\pi_{\tilde{G}_{n+\theta+\varkappa, \ell}}^k)^*(\hat{\gamma}_{\tilde{G}_{n+\theta+\varkappa, \ell}}^{\ell})|_{\Omega_{\mathcal{K}}(\gamma_{\tilde{G}_{n+\theta+\varkappa, \ell}}^{n+\theta+\varkappa}, \mathbb{R}^{p+\theta+\varkappa})}) \wedge T(\nu_P) \right) \longrightarrow [P, B_{\mathcal{D}}].$$

### 9. PROOF OF THEOREM 1.3

The development of the h-principles has been described in detail in [24]. We only refer to the Smale-Hirsch Immersion Theorem ([46], [25]), the Feit  $k$ -mersion Theorem ([22]) and the general theorems due to Gromov [23] and du Plessis ([16], [17], [18]). In particular, du Plessis has proposed a nice condition called “extensibility” under which the h-principle holds for  $\Omega^I$ -regular maps or smooth maps with only  $\mathcal{K}$ -simple singularities. However, this extensibility condition is not so effective in the dimensions  $n \geq p$ . On the other hand, Eliášberg ([20], [21]) has proved the famous h-principle on the 1-jet level for sections  $s : N \rightarrow \Omega^1(N, P)$  which have a given fold map  $f_0$  defined around  $s^{-1}(\Sigma^1(N, P))$  such that  $(j^2 f_0)^{-1}(\Sigma^{1,0}(N, P)) = s^{-1}(\Sigma^1(N, P))$  and the fold singularities of any semi-index of  $f_0$  are not empty.

In order to prove Theorem 1.3 by applying Theorems 1.1 and 1.2 we have to show that the assumption concerning h-principles is satisfied in the situation of Theorem 1.3. We have proved the h-principle in (h-P) for fold-maps in [3] and [6]. Recently we have introduced a very effective condition for the h-principle in (h-P) in [9]. As an application we can prove the following theorem by using [9] (see a proof in [10]). Let  $k \gg n, p$  as in Introduction.

**Theorem 9.1.** *Let  $n < p$  or  $n \geq p \geq 2$ . Let  $\Omega(n, p)$  denote a  $\mathcal{K}$ -invariant open subspace in  $J^k(n, p)$  such that when  $n \geq p \geq 2$ ,  $\Omega(n, p)$  contains  $\Sigma^{n-p+1,0}(n, p)$  at least. Then the h-principle in (h-P) holds for  $\Omega(n, p)$ .*

In particular, we show the following examples of  $\Omega(n, p)$ :

- (i)  $\Omega^I(n, p)$  such that when  $n \geq p \geq 2$ ,  $I \geq (n - p + 1, 0)$  ([8]),
- (ii) an open subspace consisting of all regular  $k$ -jets and a finite number of  $\mathcal{K}$ -orbits of  $\mathcal{K}$ -simple singularities such that when  $n \geq p \geq 2$ , it contains all fold jets in addition ([9]).

If  $\Omega_*(n+1, p+1) = \Omega_{\mathcal{K}}^1$ , then we write  $\mathfrak{C}(n, P; \Omega)$  simply in place of  $\mathfrak{C}(n, P; \Omega, \Omega_{\mathcal{K}})$  in the following. We have the following corollary of Theorems 1.1, 1.2 and 9.1.

**Corollary 9.2.** *Let  $n < p$ . Let  $\Omega(n, p)$  and  $\Omega_{\mathcal{K}}^1$  be the  $\mathcal{K}$ -invariant open subsets given in Theorem 1.2. Let  $P$  be a  $p$ -dimensional manifold. Then the homomorphism*

$$\omega : \mathfrak{C}(n, P; \Omega) \longrightarrow \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^1}^{\ell}) \right)$$

*is an isomorphism.*

If we apply Theorem 9.1 to [42, Theorem 9.2], then under the same assumption of Theorem 1.3, there exists an isomorphism of

$$(9.1) \quad \mathfrak{C}(n, P; \Omega) \longrightarrow \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_{\mathcal{K}}^m}^{\ell}) \right)$$

for a sufficiently large integer  $m$  also in the dimensions  $n \geq p \geq 2$  as well as  $n < p$ . This isomorphism is nothing but the composite of  $\omega$  and the map

$$\lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^1}^\ell) \right) \longrightarrow \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^m}^\ell) \right)$$

which is induced by  $T(\mathbf{b}(\hat{\gamma})^{(\Omega_K^1, \Omega_K^m)})$ .

We are now ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* If  $n < p$ , then we have

$$\mathfrak{D}(n, P; \Omega) \approx \lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^1}^\ell) \right)$$

by Corollary 9.2. If  $n \geq p \geq 2$ , then the homomorphisms

$$(9.2) \quad (T(\mathbf{b}(\hat{\gamma})^{(\Omega_K^{\theta+\kappa}, \Omega_K^{\theta+\kappa+q})}))_* : \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^{\theta+\kappa}}^\ell) \right) \longrightarrow \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^{\theta+\kappa+q}}^\ell) \right)$$

are isomorphisms for integers  $q \geq 0$  by Proposition 7.3. By Propositions 8.2, 8.3 and (9.1)  $\mathfrak{D}(n, P; \Omega)$  is isomorphic to  $[P, B_{\mathfrak{D}}]$ . This completes the proof.  $\square$

**Corollary 9.3.** *Under the same assumption of Theorem 1.3,  $\mathfrak{D}(n, P; \Omega) \otimes \mathbb{Q}$  is isomorphic to  $H_n(\Omega_K(\gamma_{G_{n+\theta+\kappa}}^{n+\theta+\kappa}, \mathbb{R}^{p+\theta+\kappa}) \times P; \mathbb{Q})$  in the dimensions  $n < p$  and  $n \geq p \geq 2$ .*

Here, let us see a relationship between the space  $B_{\mathfrak{D}}$ , the Thom-Atiyah duality in bordism and cobordism and the spaces introduced by Kazarian([31] and [32]). Let us recall the  $n$ -dimensional oriented bordism group  $\Omega_n(P)$  of maps to  $P$  and the Thom-Atiyah duality,  $\pi_{n+\ell}(MSO(\ell) \wedge P^0) \approx [P, \Omega^\ell MSO(\ell + p - n)]$  (see [11] and [15, Chap. I, 12 ]), where  $MSO$  denotes the Thom space of the universal bundle  $\gamma$  over  $BSO$  and  $\Omega^\ell X$  denotes the  $\ell$ -th iterated loop space. We have the following commutative diagram under the notation in (8.5) ( $\ell \gg n, p$ ), although we do not give a proof of the commutativity:

$$(9.3) \quad \begin{array}{ccc} \mathfrak{D}(n, P; \Omega) & \longrightarrow & \Omega_n(P) \\ \approx \downarrow & & \\ \text{Im}^{\mathfrak{D}} \left( T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_K)}) \right) (\subset \pi_{n+\ell} \left( T(\hat{\gamma}_{\Omega_K^1}^\ell) \right)) & & \downarrow \approx \\ \approx \downarrow & & \\ \pi_{n+\ell+p+\kappa} \left( T(\hat{\gamma}_{\Omega_K^2}^\ell) \wedge S^{p+\kappa}(P^0) \right) & \longrightarrow & \pi_{n+\ell+p+\kappa} (MSO(\ell) \wedge S^{p+\kappa}(P^0)) \\ \approx \downarrow & & \downarrow \approx \\ [P, B_{\mathfrak{D}}] & \longrightarrow & [P, \Omega^{\ell+n-p} MSO(\ell)], \end{array}$$

where

$$(1) \quad \Omega_K^1 = \Omega_K(\gamma_{\tilde{G}_{n+1, \ell}}^{n+1}, TP \oplus \varepsilon_P^1) \text{ and } \Omega_K^2 = \Omega_K(\gamma_{\tilde{G}_{n+\theta+\kappa, \ell}}^{n+\theta+\kappa}, \mathbb{R}^{p+\theta+\kappa}),$$

(2) the second and the bottom horizontal homomorphisms are induced from the canonical projections

$$\Omega_K(\gamma_{\tilde{G}_{n+\theta+\kappa, \ell}}^{n+\theta+\kappa}, \mathbb{R}^{p+\theta+\kappa}) \subset J^k(\gamma_{\tilde{G}_{n+\theta+\kappa, \ell}}^{n+\theta+\kappa}, \mathbb{R}^{p+\theta+\kappa}) \longrightarrow \tilde{G}_{n+\theta+\kappa, \ell} \subset BSO(\ell),$$

(3) the left vertical map is an isomorphism by Theorem 1.2, (9.1), Propositions 8.2 and 8.3 for  $n < p$  and  $n \geq p \geq 2$ .

According to [31] and [32, 2.18 Corollary and 2.8 Example], let us consider the subspace  $\Omega^{\ell+n-p} MSO(\ell)_{\Omega(n, p)}$  in  $\Omega^{\ell+n-p} MSO(\ell)$  which consists of all maps  $a : S^{\ell+n-p} \rightarrow MSO(\ell)$  such that  $a$  is smooth around  $a^{-1}(BSO(\ell))$  and that for

any point  $x \in a^{-1}(BSO(\ell))$  with  $a(x) = y$ , the  $k$ -jet of the composite of  $a : (S^{\ell+n-p}, x) \rightarrow (MSO(\ell), y)$  and a projection germ of  $(MSO(\ell), y)$  to the fiber  $(\gamma_y, y)$  lies in  $\Omega_{\mathcal{K}}(S^{\ell+n-p}, \gamma_y)$  associated to  $\Omega(n, p)$ . Then the cobordism class represented by an  $\Omega$ -regular map is mapped to the homotopy class represented by a map  $P \rightarrow \Omega^{\ell+n-p}MSO(\ell)_{\Omega(n, p)}$  in (9.3), whose corresponding map  $S^{\ell+n-p} \times P \rightarrow MSO(\ell)$  is transverse to  $BSO(\ell)$ .

## 10. FOLD-MAPS

Let  $m \geq q$ . Let  $V_{m, q}^{\text{row}}$  denote the Stiefel manifold  $(E_q \times O(m - q)) \setminus O(m)$  under the canonical bases of  $\mathbb{R}^m$  and  $\mathbb{R}^q$ , whose element is regarded as an epimorphism  $\mathbb{R}^m \rightarrow \mathbb{R}^q$  or a regular  $q \times m$ -matrix in the following. Let  $\mathcal{E} \rightarrow X$  and  $\mathcal{F} \rightarrow Y$  be vector bundles of dimensions  $m$  and  $q$  with metrics respectively. Let  $V(\mathcal{E}, \mathcal{F})$  denote the subbundle of  $\text{Hom}(\mathcal{E}, \mathcal{F})$  associated to  $V_{m, q}^{\text{row}}$ . Let  $\pi_X^V : V(\mathcal{E}, \mathcal{F}) \rightarrow X$  and  $\pi_Y^V : V(\mathcal{E}, \mathcal{F}) \rightarrow Y$  be the canonical projections respectively.

We have the actions of  $O(q) \times O(m)$  on  $V_{m+1, q}^{\text{row}}$  from the left-hand side through  $O(q)$  and from the right-hand side through  $O(m) \times 1$  respectively. The group  $O(q) \times O(m)$  also naturally acts on  $\Omega^{m-q+1, 0}(m, q)$ . In [4, Theorem 2.6] we have described the homotopy type of  $\Omega^{m-q+1, 0}(m, q)$  in terms of orthogonal groups and Stiefel manifolds, and have given a topological embedding

$$i_{V, \Omega} = i_{V, \Omega}^{m, q} : V_{m+1, q}^{\text{row}} \rightarrow \Omega^{m-q+1, 0}(m, q),$$

which is equivariant with respect to the actions of  $O(q) \times O(m)$ . Furthermore, if  $m - q + 1$  is odd, then there exists an equivariant map

$$R_{\Omega, V} = R_{\Omega, V}^{m, q} : \Omega^{m-q+1, 0}(m, q) \rightarrow V_{m+1, q}^{\text{row}}$$

such that  $R_{\Omega, V} \circ i_{V, \Omega}$  is the identity of  $V_{m+1, q}^{\text{row}}$ . In particular, we note that if  $m = q$ , then  $i_{V, \Omega} \circ R_{\Omega, V}$  is a deformation retraction of  $\Omega^{m-q+1, 0}(m, q)$ .

If  $i_{+1}(j_0^2 f)$  were defined by  $i_{+1}(j_0^2 f) = j_0^2(id_{\mathbb{R}} \times f)$ , then the following technical modification of  $i_{V, \Omega}$  and  $R_{\Omega, V}$  followed by Lemma 10.1 is unnecessary. Let  $h_t : \mathbb{R}^t \rightarrow \mathbb{R}^t$  be the map reversing the order of coordinates as  $h_t(x_1, x_2, \dots, x_{t-1}, x_t) = (x_t, x_{t-1}, \dots, x_2, x_1)$ . Define  $i_{V, \Omega} = i_{V, \Omega}^{m, q} : V_{m+1, q}^{\text{row}} \rightarrow \Omega^{m-q+1, 0}(m, q)$  and  $\mathfrak{R}_{\Omega, V} = \mathfrak{R}_{\Omega, V}^{m, q} : \Omega^{m-q+1, 0}(m, q) \rightarrow V_{m+1, q}^{\text{row}}$  by

$$i_{V, \Omega}^{m, q}(A) = i_{V, \Omega}^{m, q}(Ah_{m+1}) \cdot h_m \quad \text{and} \quad \mathfrak{R}_{\Omega, V}^{m, q}(j_0^2 f) = R_{\Omega, V}^{m, q}(j_0^2(f \circ h_m))h_{m+1}.$$

It will be easy to see  $\mathfrak{R}_{\Omega, V}^{m, q} \circ i_{V, \Omega}^{m, q} = id_{V_{m+1, q}^{\text{row}}}$ . Let  $i^{+1} : J^k(n, p) \rightarrow J^k(n+1, p+1)$  denote the map defined by  $i^{+1}(j_0^k f) = j_0^k(id_{\mathbb{R}} \times f)$ . Let  $j^{+1}, j_{+1} : V_{n+1, p}^{\text{row}} \rightarrow V_{n+2, p+1}^{\text{row}}$  denote the map defined by  $j^{+1}(A) = (1) \dot{+} A$  and  $j_{+1}(A) = A \dot{+} (1)$ , where  $\dot{+}$  denotes the direct sum of matrices. We consider the action of  $O(p) \times O(n)$  on  $V_{n+1, p}^{\text{row}}$  by  $(S, T) \cdot A = SA((1) \dot{+} T^{-1})$  for  $S \in O(p)$  and  $T \in O(n)$ .

**Lemma 10.1.** *Under the above notation, we have that  $i_{V, \Omega}^{n, p}$  and  $\mathfrak{R}_{\Omega, V}^{n, p}$  are equivariant with respect to the actions of  $O(p) \times O(n)$  and that*

$$\mathfrak{R}_{\Omega, V}^{n+1, p+1}(i_{+1}(j_0^2(f))) = j_{+1}(\mathfrak{R}_{\Omega, V}^{n, p}(j_0^2 f)) \quad \text{and} \quad i_{V, \Omega}^{n+1, p+1}(j_{+1}(A)) = i_{+1}(i_{V, \Omega}^{n, p}(A)).$$

*Proof.* For  $A \in V_{n+1,p}^{\text{row}}$ ,  $S \in O(p)$  and  $T \in O(n)$ , we have that

$$\begin{aligned}
i_{V,\Omega}^{n,p}((S, T^{-1}) \cdot A) &= i_{V,\Omega}^{n,p}(SA((1) \dot{+} T)) \\
&= i_{V,\Omega}^{n,p}(SA((1) \dot{+} T)h_{n+1}) \cdot h_n \\
&= S \cdot i_{V,\Omega}^{n,p}(Ah_{n+1}h_{n+1}((1) \dot{+} T)h_{n+1}) \cdot h_n \\
&= S \cdot i_{V,\Omega}^{n,p}(Ah_{n+1}(h_nTh_n \dot{+} (1))) \cdot h_n \\
&= S \cdot i_{V,\Omega}^{n,p}(Ah_{n+1}) \cdot (h_nTh_nh_n) \\
&= S \cdot (i_{V,\Omega}^{n,p}(Ah_{n+1}) \cdot h_n) \cdot T \\
&= (S, T^{-1}) \cdot i_{V,\Omega}^{n,p}(A),
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_{\Omega,V}^{n,p}(S, T^{-1}) \cdot (j_0^2 f) &= \mathfrak{R}_{\Omega,V}^{n,p}(S \cdot (j_0^2 f) \cdot T) \\
&= R_{\Omega,V}^{n,p}(S \cdot (j_0^2(f) \cdot T) \cdot h_n)h_{n+1} \\
&= SR_{\Omega,V}^{n,p}(j_0^2(f) \cdot (h_nh_n)T \cdot h_n)h_{n+1} \\
&= SR_{\Omega,V}^{n,p}(j_0^2(f) \cdot h_n \cdot (h_nTh_n))h_{n+1} \\
&= SR_{\Omega,V}^{n,p}(j_0^2(f \circ h_n))(h_nTh_n \dot{+} (1))h_{n+1} \\
&= SR_{\Omega,V}^{n,p}(j_0^2(f \circ h_n))h_{n+1}h_{n+1}(h_nTh_n \dot{+} (1))h_{n+1} \\
&= SR_{\Omega,V}^{n,p}(j_0^2(f \circ h_n))h_{n+1}((1) \dot{+} T) \\
&= (S, T^{-1}) \cdot \mathfrak{R}_{\Omega,V}^{n,p}(j_0^2 f).
\end{aligned}$$

By the definition of  $i_{V,\Omega}$  and  $R_{\Omega,V}$  we have that

$$R_{\Omega,V}^{n+1,p+1}(j_0^2(id_{\mathbb{R}} \times f)) = j^{+1}(R_{\Omega,V}^{n,p}(j_0^2 f)) \text{ and } i_{V,\Omega}^{n+1,p+1}(j^{+1}(A)) = i^{+1}(i_{V,\Omega}^{n,p}(A)).$$

Therefore, we have that

$$\begin{aligned}
\mathfrak{R}_{\Omega,V}^{n+1,p+1}(i_{+1}(j_0^2(f))) &= R_{\Omega,V}^{n+1,p+1}(h_{p+1}h_{p+1} \cdot j_0^2((f \times id_{\mathbb{R}}) \cdot h_{n+1}))h_{n+2} \\
&= R_{\Omega,V}^{n+1,p+1}(h_{p+1} \cdot j_0^2(id_{\mathbb{R}} \times (h_p \circ f \circ h_n)))h_{n+2} \\
&= h_{p+1}j^{+1}R_{\Omega,V}^{n,p}(j_0^2(h_p \circ f \circ h_n))h_{n+2} \\
&= (h_p R_{\Omega,V}^{n,p}(j_0^2(h_p \circ f \circ h_n))h_{n+1}) \dot{+} (1) \\
&= (R_{\Omega,V}^{n,p}(j_0^2(f \circ h_n))h_{n+1}) \dot{+} (1) \\
&= j_{+1}(\mathfrak{R}_{\Omega,V}^{n,p}(j_0^2(f)))
\end{aligned}$$

and other formula is similarly proved.  $\square$

If we provide  $\gamma_{G_m}^m$  and  $\mathcal{F}$  with metrics, then the structure group of  $J^2(\gamma_{G_m}^m \oplus \varepsilon_{G_m}^1, \mathcal{F})$  is reduced to  $O(q) \times O(m)$ . Let  $\Omega = \Omega^{m-q+1,0}(\gamma_{G_m}^m, \mathcal{F})$ . Let  $i_{V,\Omega}(\gamma_{G_m}^m, \mathcal{F}) : V(\gamma_{G_m}^m \oplus \varepsilon_{G_m}^1, \mathcal{F}) \rightarrow \Omega$  and  $\mathfrak{R}_{\Omega,V}(\gamma_{G_m}^m, \mathcal{F}) : \Omega \rightarrow V(\gamma_{G_m}^m \oplus \varepsilon_{G_m}^1, \mathcal{F})$  be the fiber maps associated to  $i_{V,\Omega}$  and  $\mathfrak{R}_{\Omega,V}$  respectively. Then

(10-i) the fiber map  $i_{V,\Omega}(\gamma_{G_m}^m, \mathcal{F})$  is a topological embedding,

(10-ii) if  $m - q + 1$  is odd, then the composition  $\mathfrak{R}_{\Omega,V}(\gamma_{G_m}^m, \mathcal{F}) \circ i_{V,\Omega}(\gamma_{G_m}^m, \mathcal{F})$  is the identity of  $V(\gamma_{G_m}^m \oplus \varepsilon_{G_m}^1, \mathcal{F})$ .

Let  $\hat{\gamma}_V^\ell$  denote the vector bundle induced from  $\hat{\gamma}_{G_m}^\ell$  over  $V(\gamma_{G_m}^m \oplus \varepsilon_{G_m}^1, \mathcal{F})$ . We note  $\hat{\gamma}_\Omega^\ell = (\mathfrak{R}_{\Omega,V}(\gamma_{G_m}^m, \mathcal{F}))^* \hat{\gamma}_V^\ell$ . Then we have the bundle maps  $\mathbf{b}_\mathfrak{R} : \hat{\gamma}_\Omega^\ell \rightarrow \hat{\gamma}_V^\ell$  and  $\mathbf{b}_i : \hat{\gamma}_V^\ell \rightarrow \hat{\gamma}_\Omega^\ell$  covering  $\mathfrak{R}_{\Omega,V}(\gamma_{G_m}^m, \mathcal{F})$  and  $i_{V,\Omega}(\gamma_{G_m}^m, \mathcal{F})$  respectively. Their associated

maps  $T(\mathbf{b}_{\mathfrak{A}})$  and  $T(\mathbf{b}_i)$  between the Thom spaces satisfies that  $T(\mathbf{b}_{\mathfrak{A}}) \circ T(\mathbf{b}_i)$  is equal to the identity of  $T(\hat{\gamma}_V^\ell)$ .

Let  $v^\Delta : V_{n+1,p}^{\text{row}} \rightarrow V_{n+2,p+1}^{\text{row}}$  denote the map sending an epimorphism  $A \in V_{n+1,p}^{\text{row}}$  to the direct sum  $A \dot{+} (1)$  of matrices  $A$  and  $(1)$ . Set

$$V_{n+1} = V(\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1, TP), \quad V_{n+2} = V(\gamma_{G_{n+1}}^{n+1} \oplus \varepsilon_{G_{n+1}}^1, TP \oplus \varepsilon_P^1),$$

$$\Omega = \Omega^{n-p+1,0}(\gamma_{G_n}^n, TP), \quad \Omega_K = \Omega^{n-p+1,0}(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1).$$

Let  $i^G$  denote the canonical classifying map  $G_n \rightarrow G_{n+1}$  of  $\gamma_{G_n}^n \oplus \varepsilon_{G_n}^1$ . Let  $V^\Delta : V_{n+1} \rightarrow V_{n+2}$  denote the fiber map associated to  $v^\Delta$  covering  $(i^G, \pi_P)$ . Let  $\Omega = \Omega^{n-p+1,0}$ . If  $n-p+1$  is odd, then we have the commutative diagram

$$\begin{array}{ccc} \Omega^{n-p+1,0}(\gamma_{G_n}^n, TP) & \xrightarrow{\Delta^{(\Omega, \Omega_K)}} & \Omega^{n-p+1,0}(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1) \\ \mathfrak{R}_{\Omega, V}^{n,p}(\gamma_{G_n}^n, TP) \downarrow & & \downarrow \mathfrak{R}_{\Omega, V}^{n+1,p+1}(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1) \\ V_{n+1} & \xrightarrow{V^\Delta} & V_{n+2} \end{array}$$

by Lemma 10.1, where the horizontal fiber maps cover  $(i^G, \pi_P)$ .

**Lemma 10.2.** *Let  $n-p+1$  be odd. The fiber map  $V^\Delta$  is a homotopy  $n$ -equivalence.*

*Proof.* Since  $v^\Delta$  and  $(i^G, \pi_P)$  are homotopy  $n$ -equivalences, it follows that  $V^\Delta$  is a homotopy  $n$ -equivalence.  $\square$

We have the bundle map  $\mathbf{b}_V : \hat{\gamma}_{V_{n+1}}^\ell \rightarrow \hat{\gamma}_{V_{n+2}}^\ell$  covering  $V^\Delta$  and its associated map  $T(\mathbf{b}_V)$  between the Thom spaces. Recalling the map  $T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_K)}) : T(\hat{\gamma}_\Omega^\ell) \rightarrow T(\hat{\gamma}_{\Omega_K}^\ell)$  we obtain the following commutative diagram

$$(10.1) \quad \begin{array}{ccc} \pi_{n+\ell}(T(\hat{\gamma}_\Omega^\ell)) & \xrightarrow{T(\mathbf{b}(\hat{\gamma})^{(\Omega, \Omega_K)})_*} & \pi_{n+\ell}(T(\hat{\gamma}_{\Omega_K}^\ell)) \\ \downarrow & & \downarrow \\ \pi_{n+\ell}(T(\hat{\gamma}_{V_{n+1}}^\ell)) & \xrightarrow{T(\mathbf{b}_V)_*} & \pi_{n+\ell}(T(\hat{\gamma}_{V_{n+2}}^\ell)), \end{array}$$

where the left vertical map and the right vertical maps are  $T(\mathbf{b}_{\mathfrak{A}})_*$  for  $(n, p)$  and  $(n+1, p+1)$  respectively. Let  $G_{n+2+\varkappa} = A_{n+2+\varkappa, \ell}$  as above.

**Proposition 10.3.** *Let  $n \geq p \geq 2$  and  $\ell \gg n, p$ . Let  $n-p+1$  be odd. Let  $\Omega = \Omega^{n-p+1,0}(\gamma_{G_n}^n, TP)$  and  $\Omega_K = \Omega^{n-p+1,0}(\gamma_{G_{n+1}}^{n+1}, TP \oplus \varepsilon_P^1)$  be as above. Then there exist homomorphisms*

$$\omega_V : \mathfrak{C}(n, P; \Omega^{n-p+1,0}) \longrightarrow \pi_{n+\ell}(T(\hat{\gamma}_{V_{n+2}}^\ell)),$$

$$\iota_V : \pi_{n+\ell}(T(\hat{\gamma}_{V_{n+2}}^\ell)) \longrightarrow \mathfrak{C}(n, P; \Omega^{n-p+1,0})$$

such that  $\omega_V \circ \iota_V$  is the identity of  $\pi_{n+\ell}(T(\hat{\gamma}_{V_{n+2}}^\ell))$ . In particular, if  $n = p$ , then  $\omega_V$  is an isomorphism.

*Proof.* (i) In the diagram (10.1) it follows from the similar argument as in Proof of Theorem 1.2 that  $T(\mathbf{b}_V)_*$  is an epimorphism. It follows from (10-i) and (10-ii) that  $T(\mathbf{b}_{\mathfrak{A}})_*$  in the diagram (10.1) for  $(n, p)$  and  $(n+1, p+1)$  are epimorphisms. By Theorem 1.1 we obtain the required homomorphisms  $\omega_V$  and  $\iota_V$ .  $\square$

Here, we prove the following refinement of [4, Theorem 0.3]. This theorem should be compared with the results in [29] and [30].

**Theorem 10.4.** *Under the assumption of Proposition 10.3 there exists a splitting epimorphism*

$$\mathfrak{C}(n, P; \Omega^{n-p+1,0}) \rightarrow \lim_{\varkappa \rightarrow \infty} \left( \lim_{\ell \rightarrow \infty} \pi_{n+\ell+\varkappa} \left( T(\hat{\gamma}_{G_{n-p+1}}^\ell) \wedge T(\nu_P) \right) \right).$$

In particular, if  $n = p$ , then this is an isomorphism.

*Proof.* The proof proceeds similarly as in the proof of Propositions 8.2 and 8.3 by replacing spaces  $\Omega_K(\xi, \mathcal{F})$  by  $V(\xi, \mathcal{F})$ . We only prove the unoriented case. Define the fiber map

$$(10.2) \quad V_{n+2} \longrightarrow V'(\gamma_{G_{n+1}}^{n+1} \oplus \varepsilon_{G_{n+1}}^1 \oplus \nu_P, TP \oplus \varepsilon_P^1 \oplus \nu_P),$$

where the last space is the subbundle of

$$\text{Hom}(\pi_{G_{n+1}}^*(\gamma_{G_{n+1}}^{n+1} \oplus \varepsilon_{G_{n+1}}^1) \oplus \pi_P^*(\nu_P), TP \oplus \varepsilon_P^1 \oplus \nu_P)$$

associated to  $V_{n+2+\varkappa, p+1+\varkappa}$  over  $G_{n+1} \times P$  and the map sends  $A \in (V_{n+2})_{(x,y)}$  to  $A \oplus id_{(\nu_P)_y}$ . We have a bundle map

$$(10.3) \quad \pi_{G_{n+1}}^*(\gamma_{G_{n+1}}^{n+1} \oplus \varepsilon_{G_{n+1}}^1) \oplus \pi_P^*(\nu_P) \longrightarrow \gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}$$

covering a classifying map  $c^+ : G_{n+1} \times P \rightarrow G_{n+2+\varkappa}$  (we may need to replace  $\ell$  with a bigger integer). We note that  $(c^+, \pi_P) : G_{n+1} \times P \rightarrow G_{n+2+\varkappa} \times P$  is a homotopy  $(n+1)$ -equivalence. By (10.2) and (10.3) we obtain a homotopy  $(n+1)$ -equivalent fiber map

$$(10.4) \quad V_{n+2} \longrightarrow V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, TP \oplus \varepsilon_P^1 \oplus \nu_P)$$

covering  $(c^+, \pi_P) : G_{n+1} \times P \rightarrow G_{n+2+\varkappa} \times P$ . The trivialization  $TP \oplus \nu_P \rightarrow \varepsilon_P^{p+\varkappa}$  induces the bundle map

$$(10.5) \quad V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, TP \oplus \varepsilon_P^1 \oplus \nu_P) \longrightarrow V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa}) \times P$$

over  $G_{n+2+\varkappa} \times P$ .

We denote, by  $V_{n-p+1, p+1+\varkappa, \ell}$ , the space which consists of all triples  $(a, b, c)$  where  $a$ ,  $b$  and  $c$  are mutually perpendicular subspaces in  $\mathbb{R}^{n+\ell+\varkappa+2}$  of dimensions  $n-p+1$ ,  $p+1+\varkappa$  and  $\ell$  respectively with  $a \oplus b \oplus c = \mathbb{R}^{n+\ell+\varkappa+2}$ . Let  $\gamma_V^{p+1+\varkappa}$  be the canonical vector bundle over  $V_{n-p+1, p+1+\varkappa, \ell}$  of dimension  $p+1+\varkappa$ .

We denote an element of  $V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})$  by  $(\alpha, h)$ , where  $\alpha \in G_{n+2+\varkappa}$  and  $h \in V((\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa})_\alpha, \mathbb{R}^{p+1+\varkappa})$ , which is regarded as an epimorphism. Then  $(\alpha, h)$  defines  $(\text{Ker}(h), \text{Ker}(h)^\perp, \alpha^\perp)$  in  $V_{n-p+1, p+1+\varkappa, \ell}$ , which is the triple of the kernel of  $h$ , the orthogonal complement of  $\text{Ker}(h)$  in  $\alpha$  and the orthogonal complement  $\alpha^\perp$ . Let  $V(\gamma_V^{p+1+\varkappa}, \mathbb{R}^{p+1+\varkappa})$  denote the principal bundle with fiber  $O(p+1+\varkappa)$  associated to  $\text{Hom}(\gamma_V^{p+1+\varkappa}, \mathbb{R}^{p+1+\varkappa})$ . We have the canonical homeomorphism

$$V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa}) \rightarrow V(\gamma_V^{p+1+\varkappa}, \mathbb{R}^{p+1+\varkappa})$$

which maps  $(\alpha, h)$  to  $h|_{\text{Ker}(h)^\perp} : \text{Ker}(h)^\perp \rightarrow \mathbb{R}^{p+1+\varkappa}$  over  $(\text{Ker}(h), \text{Ker}(h)^\perp, \alpha^\perp)$ . Let  $\mu$  denote the map

$$\mu : V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa}) = V(\gamma_V^{p+1+\varkappa}, \mathbb{R}^{p+1+\varkappa}) \longrightarrow V_{n-p+1, p+1+\varkappa, \ell}$$



defined by  $\mu((\alpha, h)) = (\text{Ker}(h), \text{Ker}(h)^\perp, \alpha^\perp)$ . Let

$$\rho : V_{n-p+1, p+1+\varkappa, \ell} \rightarrow G_{n-p+1, \ell+p+1+\varkappa}$$

be the map defined by  $\rho(a, b, c) = a$ . Then  $\rho \circ \mu$  gives a fiber bundle. Since  $\rho^{-1}(\mathbb{R}^{n-p+1} \times 0)$  is  $G_{p+1+\varkappa, \ell}$  for  $0 \times \mathbb{R}^{\ell+p+1+\varkappa}$ , we have that  $(\rho \circ \mu)^{-1}(\mathbb{R}^{n-p+1} \times 0)$  is  $V_{\ell+p+1+\varkappa, p+1+\varkappa}$ , namely  $O(\ell + p + 1 + \varkappa)/O(\ell)$ . Hence,  $\rho \circ \mu$  is a homotopy  $\ell$ -equivalence. We note that

$$(10.6) \quad (\rho \circ \mu)^*(\hat{\gamma}_{G_{n-p+1, \ell+p+1+\varkappa}}^{\ell+p+1+\varkappa}) \approx \hat{\gamma}_{V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})}^\ell \oplus \varepsilon_{V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})}^{p+1+\varkappa}.$$

We have that  $\pi_{n+\ell} \left( T(\hat{\gamma}_{V_{n+2}}^\ell) \right)$  is isomorphic to  $\pi_{n+\ell+p+\varkappa} \left( T(\hat{\gamma}_{V_{n+2}}^\ell \oplus \varepsilon_{V_{n+2}}^{p+\varkappa}) \right)$ . This is isomorphic to

$$\pi_{n+\ell+p+\varkappa} \left( T((\hat{\gamma}_{V_{n+2}}^\ell \oplus (\pi_P^V)^*(TP \oplus \nu_P))|_{V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, TP \oplus \varepsilon_P^1 \oplus \nu_P)}) \right).$$

Since setting  $G = G_{n+2+\varkappa, \ell+p}$ , we have the bundle map of

$$\hat{\gamma}_{V_{n+2}}^\ell \oplus (\pi_P^V)^*(TP \oplus \nu_P) \longrightarrow (\pi_G^V)^*(\hat{\gamma}_G^{\ell+p})|_{V(\gamma_G^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})} \times \nu_P$$

covering the canonical homotopy  $(n+1+\varkappa)$ -equivalent map

$$V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, TP \oplus \varepsilon_P^1 \oplus \nu_P) \longrightarrow V(\gamma_G^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa}) \times P,$$

by (10.5), the last group is isomorphic to

$$\pi_{n+\ell+p+\varkappa} \left( T((\pi_G^V)^*\hat{\gamma}_G^{\ell+p}|_{V(\gamma_G^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})}) \wedge T(\nu_P) \right).$$

This is isomorphic to

$$\pi_{n+\ell+p+1+2\varkappa} \left( T((\pi_G^V)^*(\hat{\gamma}_G^{\ell+p} \oplus \varepsilon_G^{1+\varkappa})|_{V(\gamma_G^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})}) \wedge T(\nu_P) \right).$$

Since we have the bundle map of  $\hat{\gamma}_{G_{n+2+\varkappa}}^\ell \oplus \varepsilon_{G_{n+2+\varkappa}}^{p+1+\varkappa}$  to  $\hat{\gamma}_G^{\ell+p} \oplus \varepsilon_G^{1+\varkappa}$  covering the canonical map  $G_{n+2+\varkappa} \rightarrow G$  again, this is isomorphic to

$$\pi_{n+\ell+p+1+2\varkappa} \left( T((\pi_{G_{n+2+\varkappa}}^V)^*(\hat{\gamma}_{G_{n+2+\varkappa}}^\ell \oplus \varepsilon_{G_{n+2+\varkappa}}^{p+1+\varkappa})|_{V(\gamma_{G_{n+2+\varkappa}}^{n+2+\varkappa}, \mathbb{R}^{p+1+\varkappa})}) \wedge T(\nu_P) \right).$$

Since  $\rho \circ \mu$  is a homotopy  $\ell$ -equivalence, it follows from (10.6) that the last group is isomorphic to

$$\pi_{n+\ell+p+1+2\varkappa} \left( T(\hat{\gamma}_{G_{n-p+1, \ell+p+1+\varkappa}}^{\ell+p+1+\varkappa}) \wedge T(\nu_P) \right),$$

which is isomorphic to

$$(10.7) \quad \pi_{n+\ell+\varkappa} \left( T(\hat{\gamma}_{G_{n-p+1}}^\ell) \wedge T(\nu_P) \right) \approx \{S^{n+\ell+\varkappa}; T(\hat{\gamma}_{G_{n-p+1}}^\ell) \wedge T(\nu_P)\}.$$

Since  $\ell$  is sufficiently large and  $\varkappa$  is any integer with  $\varkappa \geq p+3$ , we have proved the assertion.  $\square$

Now we define

$$B_V = \lim_{\ell \rightarrow \infty} C^0(S^{n-p+\ell}, T(\hat{\gamma}_{G_{n-p+1, \ell}}^\ell)).$$

By the duality map  $T(\nu_P) \wedge S^t(P^0) \rightarrow S^{p+\varkappa+t}$ , the last group in (10.7) is isomorphic to

$$\{S^{n+\ell+\varkappa} \wedge S^t(P^0); T(\hat{\gamma}_{G_{n-p+1}}^\ell) \wedge S^{p+\varkappa+t}\} \approx \{S^{n-p+\ell} \wedge P^0; T(\hat{\gamma}_{G_{n-p+1}}^\ell)\}.$$

Then we have the following proposition.

**Proposition 10.5.** *Let  $P$  be oriented and connected. Then we have the isomorphism*

$$\lim_{\ell \rightarrow \infty} \pi_{n+\ell} \left( T(\hat{\gamma}_{V_{n+2}}^\ell) \right) \longrightarrow [P, B_V],$$

where  $V_{n+2} = V(\gamma_{\tilde{G}_{n+1,\ell}}^{n+1} \oplus \varepsilon_{\tilde{G}_{n+1,\ell}}^1, TP \oplus \varepsilon_P^1)$ .

Furthermore, let  $n = p$  and  $F$  be the space defined in Introduction. Then it has been proved in [4, Proposition 4.1 and Remark 4.3] and [3] by applying Theorem 10.4 that there exists an isomorphism

$$(10.8) \quad \mathfrak{D}(n, P; \Omega^{1,0}) \approx [P, F].$$

In particular, the homotopy group  $\pi_n(F, *)$  of the  $F$ 's connected component of maps of degree 0 is isomorphic to the  $n$ -th stable homotopy group  $\pi_n^s$  by [3, Theorem 1 and Corollary 2].

Chess[14, 1.3 Corollary] has proved, in our notation, that  $\mathfrak{D}(n, \mathbb{R}^n; \Omega^{1,0})$  is isomorphic to  $\pi_n^s$ . We can prove this fact from (10.5). In fact, let  $\mathfrak{D}(n, S^n; \Omega^{1,0}; 0)$  denote the subset of  $\mathfrak{D}(n, S^n; \Omega^{1,0})$  which consists of all cobordism classes  $[f]$  such that the degree of  $f$  is 0. By applying the h-principles in (h-P) for fold-maps to  $\mathbb{R}^n$  and  $S^n$  in [6] we can prove that the inclusion  $\mathbb{R}^n = S^n \setminus \{(0, \dots, 0, 1)\} \rightarrow S^n$  canonically induces an isomorphism

$$\mathfrak{D}(n, \mathbb{R}^n; \Omega^{1,0}) \rightarrow \mathfrak{D}(n, S^n; \Omega^{1,0}; 0).$$

The detail is left to the reader.

The author proposes a problem: Let  $i_\Omega : \pi_n^s \approx \mathfrak{D}(n, S^n; \Omega^{1,0}; 0) \rightarrow \mathfrak{D}(n, S^n; \Omega)$  denote the homomorphism induced from the inclusion  $\Omega^{1,0}(n, n) \rightarrow \Omega(n, n)$ . For an element  $a \neq 0$  in  $\pi_n^s$ , we define a  $\mathcal{K}$ -invariant open subset  $U(a)$  in  $J^k(n, n)$  as the union of all  $\mathcal{K}$ -invariant open sets  $\Omega(n, n)$  such that  $i_\Omega(a) \neq 0$ . Study how the singularities in  $U(a)$  are related to  $a$ . The spaces  $B_\Omega$ 's for  $\mathfrak{D}(n, S^n; \Omega)$  will be useful. This should be compared with the theme and the problem studied in [7].

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